Common noise induces clustering in populations of globally coupled oscillators

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Abstract – Globally coupled phase oscillator populations under the action of common noise are considered. Our numerical investigations show that, if the oscillators are desynchronized through global coupling, application of weak common noise leads to the formation of clusters in such systems. At stronger noises, the transition to full noise-induced synchronization takes place. The observed behavior is general and should be important for a broad variety of applications.

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Since their introduction by Winfree [1], populations of globally coupled phase oscillators have become a Rosetta Stone in the modern theory of synchronization phenomena. Using such models, basic features of coupled oscillator systems, such as synchronization, desynchronization, clustering, and effects of heterogeneities and noise, can be investigated [2–6]. In the simplest and most frequently studied case of harmonic interactions, a population of identical phase oscillators can only form synchronous or asynchronous states. Clustering in globally coupled populations of identical phase oscillators is only possible if more complex interaction functions, including higher harmonics, are considered [4,7–10]. When oscillators are subject to the action of external noise and the noise acting on each individual oscillator is independent, fluctuations tend to destroy synchronization and it can only persist if the coupling is strong enough [2,11–13]. In real-world situations, however, it is often found that all oscillators in a population are simultaneously affected by the same common noise. This should be, for example, the case if one of the common parameters of all oscillators, such as the temperature or the chemical concentration, undergoes random temporal variations. It has been demonstrated that, if the oscillators are not coupled, the action of common noise induces synchronization in the population [14–16]. It has also been shown that entire neural networks can be synchronized by common noise [17].

The question which we address in our present study is what would be the action of common noise on a globally coupled population of phase oscillators. Obviously, if coupling already induces synchronization, this behavior can only be enhanced by the common noise. But what happens if the interactions between the oscillators instead favor desynchronization in this system?

Below we show that, even in the classical case of harmonic interactions, application of common noise generally induces clustering in globally coupled populations. For identical oscillators, clustering sets in without a threshold, at any noise intensity. As the noise gets stronger, it is replaced by complete synchronization, characteristic for the population of independent oscillators subject to common noise. Only the regimes with two or three clusters could be found in our numerical simulations. The effect of clustering induced by common noise has been also observed in a heterogeneous population with a dispersion of natural frequencies. In this latter case, the common noise should exceed a threshold in order that the clustering takes place.

In this letter, we consider a population of oscillators whose phases \(\phi_i\) evolve with time according to a set of equations \((i = 1, 2, \ldots, N)\)

\[
\frac{\partial \phi_i}{\partial t} = \omega + \frac{1}{2i} \left( Z e^{2\pi i \alpha} e^{-i\phi_i} - c.c. \right) + \frac{\sigma}{2i} \left( \xi(t) e^{-i\phi_i} - c.c. \right). \tag{1}
\]
Thus, each oscillator interacts with the global field \( Z(t) \) collectively generated by all oscillators,

\[
Z \equiv \frac{1}{N} \sum_{i=1}^{N} e^{i\phi_i}. \tag{2}
\]

The interaction is characterized by the phase shift \( 2\pi \alpha \). In addition, all oscillators are subject to the same common noise \( \sigma \xi(t) \). Here, \( \xi(t) \) is a Gaussian white noise with \( \langle \xi(t) \rangle = 0 \), \( \langle \xi(t) \xi(t + \tau) \rangle = 0 \) and \( \langle \xi(t) \xi^*(t + \tau) \rangle = 2\delta(\tau) \). The parameter \( \sigma \) is used to specify the noise intensity. It can be shown that this model is obtained as a phase reduction for a population of globally coupled Stuart-Landau oscillators with a noise term that is statistically isotropic in the complex plane, with independent real and imaginary parts.

Note that, equivalently, the model can be written as

\[
\frac{\partial \phi_i(t)}{\partial t} = \omega + \frac{1}{N} \sum_{j \neq i} \sin(\phi_j - \phi_i + 2\pi \alpha) + \sigma |\xi(t)| \sin(\arg(\xi(t)) - \phi_i). \tag{3}
\]

Because the natural frequencies of all oscillators are identical (and equal to \( \omega \)), the terms with \( \omega \) in the model can be eliminated by going to a co-rotating reference frame, as always assumed below.

In the absence of noise (\( \sigma = 0 \)), the system always has a unique stationary state corresponding to full synchronization (\( |Z| = 1 \)) and a family of stationary asynchronous states with \( Z = 0 \). For \( 0 < \alpha < 1/4 \), the synchronous state is stable and all asynchronous states are unstable. For \( \alpha > 1/4 \), the situation is opposite: the synchronous state is unstable and all asynchronous states are attracting fixed points of the system, which are all neutrally stable. The bifurcation at \( \alpha = 1/4 \) consists of the exchange of stability between the synchronous and the asynchronous states. The bifurcation scenario is symmetric with respect to \( \alpha = 1/2 \).

We have numerically investigated the dynamical properties of the system (1) for different interaction parameters \( \alpha \) and various noise intensities \( \sigma \). Typically, our simulations have been performed for a population of size \( N = 100 \). Figures 1(b)–(d) present the results of a typical simulation. Here, snapshots of phase distributions in the system at three different time moments are shown. Initially (at \( t = 0 \)), all phases were randomly chosen. Two clusters were formed \( (t = 25000) \) and, as time goes on, they tended to be increasingly narrow (see, \( t = 50000 \)), approaching delta distributions.

To characterise the long-time distributions of phases, we monitored the phase distance between the elements, defined for elements \( i \) and \( j \) as \( d_{i,j} = \min[|\phi_j - \phi_i|, 2\pi - |\phi_j - \phi_i|] \). With this, we introduce the mean distance between close elements as

\[
q \equiv \frac{1}{N^2} \sum_{i,j=1}^{N} d_{i,j} \quad \text{with} \quad d_{i,j} = \min[|\phi_j - \phi_i|, 2\pi - |\phi_j - \phi_i|]. \tag{4}
\]

\[
\rho_k \equiv \frac{1}{N} \sum_{j=1}^{N} \exp(ik\phi_j). \tag{5}
\]

The first of them \( (k = 1) \) coincides with the global field \( Z \) and specifies the degree of synchronization. When \( |\rho_1| = 1 \), complete synchronization is reached. The second order parameter \( \rho_2 \) is sensitive to the presence of two clusters with opposite phases in the circle. If \( |\rho_2| = 1 \), but \( |\rho_1| = 0 \), two oppositely located clusters with equal sizes are formed, which include the entire population. Additionally, we have \( |\rho_2| = 1 \) in the fully synchronous state with \( |\rho_1| = 1 \).
Common noise induces clustering in populations of globally coupled oscillators

Fig. 3: (Colour on-line) Average magnitude of the first two order parameters (a) and average phase distance between the clusters (b) as a function of the noise intensity. $N = 100$, $\alpha = 0.3$.

Generally, if the $k$-th order parameter reaches $|\rho_k| = 1$ and all lower order parameters vanish, a state with $k$ equally spaced clusters containing the entire population, is established.

In fig. 3(a), we display the average moduli of the first two order parameters in the long-time limit as a function of the noise intensity $\sigma$ for $\alpha = 0.3$ (note that the logarithmic scale for the variable $\sigma$ is chosen here). For strong noises, the system is in the fully synchronous state, whereas for weak noises it is in the state with two opposite clusters. At intermediate noise intensities, a transition from two clusters to one cluster takes place. In this intermediate region, both order parameters acquire comparable values. These values do not mean a phase dispersion in the system. The system always forms just one or two asymptotically singular phase clusters. When two such clusters are formed, they may be, however, not equally spaced. Obviously, if only two point-clusters are present, but they are separated by a distance less than $\pi$, such a state would be characterized by nonzero values of both $\rho_1$ and $\rho_2$. Note that the relative phase positions of the two clusters never remain fixed; they fluctuate depending on the actual noise realization. Figure 3(b) shows how the mean distance between the two clusters depends on the noise intensity $\sigma$. It can be seen that this mean distance rapidly decreases near a certain critical noise intensity, indicating the fusion of two separate clusters into a single one. Thus, the clusters do not become destroyed in such a transition. Rather, they progressively approach one another, on the average, until the fusion occurs.

The cluster organization of the system depends on the interaction constant $\alpha$. Figure 4 displays the dependence of the first three mean order parameters on this constant for a fixed noise intensity $\sigma$ in the long-time limit. In agreement with the stability analysis, the one-cluster state, corresponding to full synchronization is observed inside the interval $0 < \alpha < 1/4$. As $\alpha$ is increased above $1/4$, it is replaced by the two-cluster state, which exists for $1/4 < \alpha < \alpha_c$ with $\alpha_c \approx 0.375$. Finally, in the interval $\alpha_c < \alpha < 1/2$ the three-cluster state is found. The video video1b.mpg provided as supplementary information illustrates the development of a three-cluster state starting from random initial conditions (see also fig. 2). Note that the convergence time to the states with three clusters is much longer than that for the two-cluster states.

In our numerical simulations we could not observe the formation of states with more than three clusters. When such states were chosen as initial conditions, they eventually transformed into two- or three-cluster states, with very long transients. Furthermore, the asymptotically formed clusters tended to have equal sizes, and states with the clusters of equal weights were typically approached even if nonequal clusters have been chosen as initial conditions. When $N = 60$, three equal clusters can be formed and they were always seen in the simulations for $\alpha_c < \alpha < 1/2$. For $N = 100$, such configuration is not however realizable and, in the simulations, we saw that some wandering of individual oscillators between the clusters of approximately equal size persisted even in the long-time limit. A similar situation has been encountered when the total number of elements was odd and two clusters tended to form. It should be however noted that in some simulations carried under relatively strong noise, in the transition region to full synchronization, non-equal clusters persisted within the entire simulation time. We cannot exclude the possibility that they corresponded to asymptotical states with different cluster weights.

We did not observe hysteresis in the transition from the two- to the three-cluster states. However, the transients associated with such a transition were different depending on its direction. If we started with the two-cluster
state and increased \( \alpha \) beyond \( \alpha_c \), we observed that the two-cluster state was first destroyed, with the oscillators getting broadly dispersed. After that, they rearranged, gradually forming three equal clusters. On the other hand, if we went in the opposite direction and started with the three-cluster state, the behavior was qualitatively different. The three clusters persisted during the entire transition and the transition was gradual. Some elements were sporadically leaving one of the clusters and joining the other two. As a result, this cluster gradually evaporated and finally disappeared.

Above, all elements were taken to be identical, so that their natural frequencies were equal. To investigate the robustness of the clustered states against structural perturbations, we have additionally performed some simulations for heterogeneous oscillator populations. In this case, the system is described by a set of equations

\[
\frac{\partial \phi_i}{\partial t} = \omega_i + \frac{1}{2i} (Z e^{2\pi i \alpha} e^{-i \phi_i} - \text{c.c.}) + \frac{\sigma}{2i} (\xi(t) e^{-i \phi_i} - \text{c.c.}),
\]

where individual frequencies \( \omega_i \) are drawn at random from a Gaussian distribution with \( \langle \omega_i \rangle = 0 \) and the dispersion \( \langle \omega_i^2 \rangle = \delta_{\omega} \). Figure 5 shows the dependence of the average magnitude of the first three order parameters on the frequency dispersion \( \delta_{\omega} \) in the parameter region where two clusters are formed in the uniform system. The two clusters persist for weakly heterogeneous populations, but are destroyed as the statistical dispersion of natural frequencies is increased.

Similar behavior has been found when the oscillators were identical, but random independent noises were acting on each phase oscillator. The destruction of the synchronization induced by common noise in the presence of sufficiently strong independent noises has been previously discussed for oscillator populations without global coupling [14–16].

While the investigations reported in the present letter were all carried out for a system of \( N = 100 \) oscillators, simulations for other systems sizes, varying from a few elements to large populations with thousands of elements, have also been performed. No principal dependence on the system size has been detected. For instance, a population of 6 identical oscillators evolves, under the action of common noise, towards the states with one, two or three clusters, depending on the noise intensity and interaction parameters, similar to what has been seen for the system of 100 oscillators.

Summarizing, we have found that the action of weak common noise on asynchronous globally coupled oscillator populations generally leads to clustering in such systems. This new knowledge, gained through numerical simulations, can have important practical implications. In many experimental situations, individual oscillators, which may represent biological cells or chemical reactors, are intrinsically subject to small random variations of common system parameters, such as temperature, illumination, etc. On the other hand, experimentally monitored properties of such oscillator simulations are often only sensitive to the degree of full synchronization in the system, specified by the global field \( Z \). When the global field vanishes or reaches only small values, this would be usually interpreted as an indication of an asynchronous state. As we have seen, however, weak common noises would typically transform asynchronous states with random oscillator phases into clustered — and hence well-organized — configurations. It may therefore be that some of the previously reported asynchronous states were actually clustered.

This letter reports the first results of our numerical investigations. The analytical theory of these phenomena is under development and shall be published separately. Note that the theory [14–16] available for the action of common noise on independent oscillators cannot be directly applied to the considered system because the phases of the coherent clusters are not fixed, but fluctuate in time, so that the positions of clusters and distances between them are not asymptotically constant.

Generally, we would like to note that oscillator populations with the desynchronization imposed by global coupling represent very interesting dynamical systems. Any phase configuration, such that the total field \( Z = 0 \), obviously represents a stationary state of such a system. The condition \( Z = 0 \) defines an \((N-2)\)-dimensional hypersurface (with \( \sum_{i=1}^{N} e^{i \phi_i} = 0 \)) in the \( N \)-dimensional phase space of the oscillator population. Any states on this hypersurface are marginally stable against the perturbations leaving the system on this hypersurface, so that only the transversal perturbations are suppressed. The role of the common noise is to “kick off” the dynamical system away from the hypersurface, unfreezing its internal dynamics, which can favor the formation of various coherent states.

Fig. 5: Average values of the magnitudes of the order parameters as a function of the variance \( \delta_{\omega} \) of the natural frequencies. \( N = 100, \alpha = 0.30 \) and \( \sigma = 0.1 \).
Common noise induces clustering in populations of globally coupled oscillators

Instead of global coupling, corresponding to some intrinsic interactions between the oscillators, systems with global feedback can furthermore be considered. If the global feedback is delayed, its action on an oscillator population is similar \[18,19\] to the effect of global coupling with the phase shift \( \alpha = \omega \tau \), where \( \tau \) is the delay time. Based on the results of present investigations, we expect that the application of global delayed feedbacks to oscillator populations subject to external common noise shall induce phase clustering in such systems, with the emerging clustering configuration dependent on the delay time. This effect can be also tested in experiments.

While only globally coupled periodic phase oscillators have been considered here, common noise can also act on the populations of coupled chaotic oscillators. Previously, it has been shown that application of common noise may enhance phase synchronization of two coupled Rössler oscillators \[20\] and of a lattice of locally coupled Rössler oscillators \[21\]. We conjecture that the effects of dynamical clustering induced by common noise should be also possible for chaotic oscillators. To observe them, global coupling between the elements must be introduced in such a way that it acts against the synchronization and the common noise should be applied to all oscillators.

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REFERENCES


