Self-organized pacemakers in birhythmic media

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Abstract

A birhythmic dynamical system is characterized by two coexisting stable limit cycles. In this article, a general reaction–diffusion system close to a supercritical pitchfork–Hopf bifurcation is investigated, where a soft onset of birhythmicity is possible. We show that stable self-organized pacemakers, which give rise to target patterns, exist and represent a generic type of spatio-temporal patterns in such a system. This is verified by numerical simulations which also show the existence of breathing and swinging pacemaker solutions. Stable pacemakers inhibit the formation of other pacemakers in the system. The drift of self-organized pacemakers in media with spatial parameter gradients is analytically and numerically investigated. Furthermore, instabilities induced by phase slips are also considered.

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1. Introduction

Oscillatory reaction–diffusion systems, such as the Belousov-Zhabotinsky (BZ) reaction, exhibit a rich variety of nonlinear wave patterns [1]. The first complex pattern discovered in this reaction was the target pattern, where concentric waves were emitted by a pacemaker representing a periodic wave source [2]. Similar target structures have been observed in many other chemical, physical, and biological systems (see, e.g. [3–7]). A simple theoretical explanation of target patterns in oscillatory chemical systems is that the pacemakers are created by impurities which locally modify the kinetic properties of the medium such that the oscillation frequency is increased [8]. Indeed, most of the target patterns seen in the BZ reaction are caused by small local inhomogeneities, such as dust particles [9–11]. However, there are also experimental observations where pacemakers could not be related to any impurities [12]. A general question is whether self-organized pacemakers, representing an intrinsic dynamical property, are possible in the absence of heterogeneities. Examples of such pacemakers with localized or extended wave patterns are known for several models [13–21]. An approximate analytical solution for self-organized pacemakers in a system, where uniform oscillations were unstable, has been constructed [14]. Furthermore, self-organized target patterns

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were found near a Hopf bifurcation with a finite wavenumber where uniform oscillations of the medium were absolutely unstable [20]. The model [15,16] was constructed to explain target pattern formation in electrohydrodynamic convection and was based on a Hopf bifurcation of a cellular spatial structure. Ohta et al. [19] have investigated a two-component activator–inhibitor model with coexistence of excitable kinetics and stable uniform oscillations, and reported several different kinds of self-organized wave sources. Stable patterns resembling localized target patterns are also found in the quintic complex Ginzburg–Landau equation [17] where stable breathing and moving oscillating objects also have been observed [22].

In the present study we consider self-organized pacemakers in birhythmic media, which are oscillatory systems where two stable limit cycles corresponding to uniform oscillations with different frequencies coexist. The phenomenon of birhythmicity has been discussed in chemical and biochemical systems [23–27], including such important examples as glycolytic oscillations [26], calcium oscillations [28], and the photosensitive BZ reaction [29]. One possible bifurcation which can give rise birhythmicity is the supercritical pitchfork–Hopf bifurcation. Near such a bifurcation a general description of spatio-temporal behavior in terms of amplitude equations is possible. In a recent Letter [30] we have shown that in a system, governed by these equations, self-organized pacemakers with extended wave patterns are generic and can be stable.

In this article, an approximate analytical solution for self-organized pacemakers in one-dimensional birhythmic reaction–diffusion systems near a pitchfork–Hopf bifurcation is constructed. The existence of self-organized pacemakers in such media is verified by simulations and their dynamical properties are numerically investigated. After introducing the model in Section 2, we derive the analytical solution for self-organized pacemakers in Section 3. We also describe their drift caused by spatial parameter gradients. Results of our numerical simulations are presented in Section 4. We confirm the existence and stability of self-organized pacemakers in one and two spatial dimensions. Then, the interaction between pacemakers is analyzed and we demonstrate that stable pacemakers globally inhibit the formation of other pacemakers in the system. Furthermore, kinetic instabilities leading to breathing and swinging pacemakers as well as instabilities induced by phase slips are investigated numerically. The article ends with a discussion of the obtained results.

2. The model

Our model consists of two coupled amplitude equations valid in the vicinity of a supercritical pitchfork–Hopf bifurcation. These equations describe an oscillatory medium near a soft onset of birhythmicity. Our analysis presented in Section 3 is based on the phase dynamics approximation of this model. Before we proceed to that derivation, a qualitative discussion of self-organized pacemakers is given.

2.1. Amplitude equations for the distributed pitchfork–Hopf bifurcation

The derivation of normal forms for reaction–diffusion systems near to certain bifurcations has recently been discussed by one of the authors [31,32]. In this article, we consider a system near a bifurcation where a stationary uniform state becomes unstable due to the simultaneous growth of a real uniform eigenmode and a pair of complex conjugate uniform eigenmodes. The generic bifurcation for this case is the fold–Hopf bifurcation. If additionally pitchfork symmetry is assumed, the bifurcation scenario is governed by a supercritical pitchfork–Hopf bifurcation. In the vicinity of this bifurcation, two stable limit cycles with different frequencies coexist, which means that the medium is birhythmic.

Fig. 1 illustrates how the combination of a supercritical pitchfork and Hopf bifurcations yields birhythmicity. If we start from the stable fixed point (shown in the upper left part of the figure) and increase the control parameter
corresponding to the Hopf bifurcation, we see that the stable fixed point is transformed into a stable limit cycle. On the other hand, if we vary the parameter responsible for the pitchfork bifurcation, the original stable fixed point splits into two new stable fixed points. If both parameters are changed simultaneously, two stable limit cycles with slightly different frequencies and amplitudes are obtained.

The normal form of the pitchfork–Hopf bifurcation is [33]

\[
\begin{align}
\dot{w} &= \eta_1 w + g_0 w z + g_1 |w|^2 w + g_2 w^2, \\
\dot{z} &= \eta_2 + \eta_3 z + c_0 |w|^2 + c_2 |w|^2 z + c_3 z^3,
\end{align}
\]

where \(w\) is the complex oscillation amplitude, \(z\) the amplitude of the real mode, and \(\eta_1, \eta_3\) the Hopf and pitchfork bifurcation parameters, respectively. The parameter \(\eta_2\) accounts for the imperfection of the pitchfork bifurcation, leading to a cusp scenario [34]. The coefficients \(g_0, g_1,\) and \(g_2\) are complex whereas \(c_0, c_2,\) and \(c_3\) are real. For positive \(\eta_1, \eta_2,\) and \(\eta_3\) there is a region in parameter space where birhythmicity is realized. Fig. 2 shows this region.

Fig. 2. The cusp in the parameter subspace spanned by \(\eta_2\) and \(\eta_3\). The gray-shaded area denotes the region where birhythmicity is found.
in the parameter space spanned by \( \eta_2 \) and \( \eta_3 \). The bifurcation scenario contained in Eqs. (1a) and (1b) is described in detail in [33].

In this article we consider reaction–diffusion systems near to the supercritical pitchfork–Hopf bifurcation. After introducing the diffusion terms into Eqs. (1a) and (1b) and performing an appropriate rescaling of spatial coordinates, time, and the amplitude variables we obtain the following amplitude equations

\[
\begin{align*}
\dot{A} &= A - (1 + i \omega)|A|^2 A + (1 + i \beta) \nabla^2 A + (1 - i \epsilon) \nabla^2 z, \\
\tau \dot{z} &= \sigma - \gamma |A|^2 + z - \nu z^3 + l^2 \nabla^2 z,
\end{align*}
\]

where \( A \) is the rescaled complex oscillation amplitude and \( z \) denotes the rescaled amplitude of the real mode. The parameters \( \alpha \) and \( \beta \) determine the nonlinear frequency shift and wave dispersion, respectively. The parameter \( \epsilon \) specifies the frequency shift of the oscillations due to the coupling to the real mode, \( \gamma \) characterizes the strength of the feedback from the oscillatory to the real mode, and the positive parameter \( \nu \) determines the nonlinear saturation of the amplitude of the real mode. The parameters \( \tau \) and \( l \), respectively, describe the ratios of time and length scales of the real and oscillatory modes. The term \( \sigma - \gamma |A|^2 \) is responsible for the imperfection of the pitchfork bifurcation. Besides of the constant parameter \( \sigma \) this term also has a space- and time-dependent contribution \( \gamma |A|^2 \) which represents the coupling to the oscillatory mode. This term will turn out to be important for the formation of stable self-organized pacemakers. The positive coefficient \( \nu \) determines nonlinear saturation of the amplitude of the real mode. Note that terms proportional to \( w z^2 \) and \( |w|^2 z \), which are present in the normal form of a pitchfork–Hopf bifurcation, have been dropped in the rescaled amplitude equations (2a) and (2b). Their contribution is negligible if \( \nu \gg 1 \) as assumed throughout this article.

The system (2a) and (2b) represents the model investigated in this article. It can be viewed as consisting of the complex Ginzburg–Landau equation (CGLE) [35] coupled to an equation for a variable exhibiting bistable dynamics.

Eq. (2a) is invariant with respect to global phase shifts \( A \rightarrow A \exp(i \phi) \). The transformation \( (A, \alpha, \beta, \epsilon) \rightarrow (A^*, -\alpha, -\beta, -\epsilon) \) leaves the equations unchanged, so that only half of the parameter space spanned by \( \alpha, \beta \), and \( \epsilon \) needs to be investigated, i.e., the sign of one parameter can be fixed arbitrarily.

### 2.2. Birhythmicity

The system (2a) and (2b) has two coexisting solutions corresponding to stable uniform oscillations. It is convenient to write the complex oscillation amplitude \( A \) in terms of the oscillation phase \( \phi \) and the real oscillation amplitude \( \rho \) defined by \( A = \rho \exp(-i \phi) \). Then, Eqs. (2a) and (2b) are transformed into

\[
\begin{align*}
\dot{\rho} &= (\rho - \rho^3) \rho, \\
\dot{\phi} &= \rho \phi^2 + \epsilon \rho, \\
\tau \dot{z} &= \sigma - \gamma \rho^2 + z - \nu z^3.
\end{align*}
\]

For harmonic limit cycle oscillations the amplitudes \( \rho \) and \( z \) are constant and related through

\[
\rho = \sqrt{\frac{z + 1}{\nu}}
\]

as follows from Eq. (3a). Substituting this into (3c) we obtain the cubic equation

\[
\nu z^3 - (1 - \gamma) z = \sigma - \gamma.
\]

Since the amplitudes of the limit cycles are completely determined by the stationary solutions of \( z \), birhythmicity occurs if \( z \) exhibits bistability. For this case, the three roots of Eq. (5) have to be real, which is fulfilled when
4(γ − 1)^3 + 27ω(γ − 1)^2 < 0. Since ω is positive, this means that the condition γ < 1 is necessary. The roots \( z_1 < z_2 < z_3 \) are found using the Cardan formula and are given by

\[
\begin{align*}
z_1 &= 2 \nu_{\phi}^{1/3} \cos \left( \frac{1}{3} \phi_{\gamma} + 2 \pi \right), \\
z_2 &= 2 \nu_{\phi}^{1/3} \cos \left( \frac{1}{3} \phi_{\gamma} + 4 \pi \right), \\
z_3 &= 2 \nu_{\phi}^{1/3} \cos \left( \frac{1}{3} \phi_{\gamma} \right),
\end{align*}
\]

where \( \nu_{\phi}^{2/3} = (1 − γ)/3ν \) and \( \cos \phi_{\gamma} = (σ − γ)/2νϕ \). The roots \( z_1 \) and \( z_3 \) represent the stable stationary solutions of \( z \) while \( z_2 \) describes the unstable stationary solution of \( z \).

The frequency of uniform oscillations is determined by \( \omega = ϕ \) and can be obtained directly from Eqs. (3b) and (4) as

\[
\omega_{1,2,3} = ω + (α + ϵ)z_{1,2,3}.
\]

The typical solutions of such an equation are fronts connecting two stable uniform states. The presence of waves with non-vanishing

2.3. Phase dynamics approximation

In terms of phase and amplitude, the system described by Eqs. (2a) and (2b) reads

\[
\begin{align*}
\dot{\varrho}_1 &= (z + 1 - ρ^2)ρ + \nabla^2 ρ - ρ(\nabla ϕ)^2 + β_0 \nabla^2 ϕ + 2β \nabla ϕ \nabla ρ, \\
\dot{\varrho}_2 &= αρ^2 + εz + \left( \frac{2}{ρ} \right) \nabla^2 ρ, \\
\tau \dot{z} &= σ - γρ^2 - z - νz^3 + l^2 \nabla^2 z.
\end{align*}
\]

If phase perturbations are sufficiently smooth, i.e., if \( |\nabla ϕ| \ll 1 \), small amplitude perturbations of a stable limit cycle decay much faster than phase perturbations [36]. As a result, the amplitude \( ρ \) is approximately given by

\[
\dot{\varrho}^2 \approx 1 + z - (\nabla ϕ)^2 + β \nabla^2 ϕ,
\]

and is adiabatically eliminated from Eqs. (8a)–(8c), which then become

\[
\begin{align*}
\dot{\varrho}_1 &= α + (α + ε)z + (β - α)(\nabla ϕ)^2 + (1 + αβ) \nabla^2 ϕ, \\
\tau \dot{z} &= σ - γ(\nabla ϕ)^2 - βγ \nabla^2 ϕ + (1 - γz) - νz^3 + l^2 \nabla^2 z.
\end{align*}
\]

In Eq. (10a) we have neglected two terms proportional to \( \nabla^2 z \) and \( Vz \). The conditions under which this is justified are discussed in Section 3.3.

The phase dynamics equations (10a) and (10b) constitute the basis of the analytical derivation in Section 3. Eq. (10b) can be interpreted as describing the reaction and diffusion of a bistable component \( z \). The typical solutions of such an equation are fronts connecting two stable uniform states. The presence of waves with non-vanishing
Fig. 3. Schematic view of a self-organized pacemaker.

phase gradients modifies the motion of these fronts. If the variable \( z \) is fixed, Eq. (10a) reduces to a standard phase dynamics equation for the CGLE. Uniform oscillations in this system become unstable if the phase diffusion coefficient \( 1 + \alpha \beta \) is negative. Throughout this article, it is assumed that uniform oscillations are modulationally stable, i.e., the Benjamin–Feir–Newell criterion \( 1 + \alpha \beta > 0 \) is fulfilled.

2.4. Self-organized pacemakers

Though a large variety of spatio-temporal patterns is described by Eqs. (10a) and (10b), we shall here focus specifically on solutions representing self-organized pacemakers. Fig. 3 shows a schematic illustration of the pacemaker solution. The variable \( z \) is increased inside the central region of the pacemaker, which is called the core.

Suppose that this spatial distribution of \( z(x) \) is fixed. According to Eq. (10a), the wave pattern generated by such an inhomogeneity, is described within the phase dynamics approximation by

\[
\partial_t \phi = \omega(x) + (\beta - \alpha)(\nabla \phi)^2 + (1 + \alpha \beta) \nabla^2 \phi,
\]

(11)

where \( \omega(x) = \alpha + (\alpha + \epsilon)z(x) \). This means that, with respect to the oscillatory subsystem, a local heterogeneity is created, which approximately obeys

\[
\omega(x) = \begin{cases} 
\omega_{\text{out}} & \text{for } |x| > R, \\
\omega_{\text{out}} + \Delta \omega & \text{for } |x| \leq R,
\end{cases}
\]

(12)

where \( \omega_{\text{out}} = \alpha + (\alpha + \epsilon)z_{\text{out}} \) is the frequency outside the core and \( \Delta \omega = (\alpha + \epsilon)(z_{\text{center}} - z_{\text{out}}) \) corresponds to the frequency shift present inside the core region of width \( 2R \).

One special solution of Eq. (11) is well known (see [36–38]): Outside the core region, plane waves \( A(x, t) = \sqrt{1 - k^2} \exp(ikx - i\omega_k t) \) are propagating and a one-dimensional target pattern is formed. The frequency \( \omega_k \) of these waves depends on their wavenumber \( k \) and is determined by the dispersion relation

\[
\omega_k = \omega_{\text{out}} + (\beta - \alpha)k^2.
\]

(13)

The factor \( \beta - \alpha \) is called the dispersion coefficient since it determines whether the oscillation frequency increases or decreases as a function of the wavenumber of the waves.
Usually, just the case $\beta - \alpha > 0$ is considered, which means that the medium has positive dispersion. In this case, a pacemaker is realized only if the local oscillation frequency is increased inside the core ($\Delta \omega > 0$). Consequently, the condition $(\alpha + \epsilon)(z_{\text{center}} - z_{\text{out}}) > 0$ must be fulfilled. Since we have assumed that $z$ is increased inside the core, this implies that $\alpha + \epsilon$ must be positive. Obviously, if $\alpha + \epsilon < 0$, the variable $z$ should instead exhibit a local decrease inside the core.

The case of negative dispersion $\beta - \alpha < 0$ is also possible, but is, in general, less investigated. In this case, a local frequency decrease inside the core ($\Delta \omega < 0$) gives rise to the formation of the target pattern, which implies that the condition $(\alpha + \epsilon)(z_{\text{center}} - z_{\text{out}}) < 0$ must be fulfilled. When $\alpha + \epsilon < 0$, the variable $z$ should increase locally when $\alpha + \epsilon > 0$. A striking property of the target waves in media with negative dispersion is that they move towards the center instead of outwards as in media with positive dispersion [37]. Recently, spiral and target waves moving towards the center of the pattern have been reported [39,40] and general properties of heterogeneous pacemakers in systems with negative dispersion have been discussed [38] (see also [41]).

To summarize, the condition for the wave emission by self-organized pacemakers can be written compactly as

$$(\beta - \alpha)(\alpha + \epsilon)(z_{\text{center}} - z_{\text{out}}) > 0.$$  

In the following, we present a detailed description of the pacemaker formation for the specific case where $z$ increases inside the core region (as sketched in Fig. 3) and the parameters obey the conditions $\beta - \alpha > 0$ and $\alpha + \epsilon > 0$.

Inside the core, the medium is therefore found in the state with a higher natural frequency. In this region, waves are initiated which propagate out of the core, giving rise to the formation of a target pattern. To create a self-organized pacemaker, a sufficiently strong perturbation, leading to a local increase of the variable $z$, should be applied. This perturbation grows accompanied by a simultaneously increase of the wavenumber $k$. The left and right boundaries of the core are composed of fronts. If their velocity $V$ is positive, the core expands, otherwise it contracts. A steady pacemaker is realized when $V = 0$. Since the front velocity $V$ depends on the wavenumber $k$ of the emitted waves and decreases for higher wavenumbers, a feedback loop is present: When a critical wavenumber is reached, the front velocity vanishes and a stationary pacemaker is formed. In contrast to usually considered pacemakers, self-organized pacemakers exist in uniform media, in the absence of any physical heterogeneities.

### 3. Analytical solution

In this section, we construct the analytical solution of a self-organized pacemaker in a one-dimensional system. Our approach is similar to a derivation for a different model system [14] where birhythmicity, however, was absent. First, the wave pattern corresponding to a fixed core is determined and the wavenumber of emitted waves is found. Second, the equation that determines the core front velocity as a function of the wavenumber is derived. Both results are then combined to obtain the properties of the stationary pacemaker and the conditions that ensure existence and stability of the pacemaker solution. For definiteness, the solution is constructed assuming that conditions $\beta - \alpha > 0$ and $\alpha + \epsilon > 0$ are satisfied. Then, the variable $z$ is increased inside the core, i.e., $z_{\text{center}} > z_{\text{out}}$. We also discuss the other parameter regimes where self-organized pacemakers are possible. Finally, the motion of pacemakers in a spatial parameter gradient is considered and the resulting drift velocity is estimated.

#### 3.1. Wave emission

Inside the core of a self-organized pacemaker, the variable $z$ is increased (see Fig. 3). Now we assume that the distribution of $z$ is known and calculate the properties of the wave pattern created by it. By applying the Hopf–Cole
transformation [36]
\[ \varphi = \frac{1 + \alpha\beta}{\beta - \alpha} \ln Q \]  
(15)
to the phase equations (10a) and (10b), the following equation for the new variable \( Q(x,t) \)
\[ \frac{\partial}{\partial t} Q = (\alpha + \epsilon)(\beta - \alpha) Q + \frac{1}{1 + \alpha\beta} z Q + (1 + \alpha\beta) \nabla^2 Q \]  
(16)
is obtained. This equation is linear and can therefore be solved exactly. It can be written as
\[ -\frac{\partial}{\partial t} Q = \hat{H} Q, \]  
(17)
with the linear operator \( \hat{H} \) defined as
\[ \hat{H} = -\frac{(\alpha + \epsilon)(\beta - \alpha)}{1 + \alpha\beta} z - (1 + \alpha\beta) \nabla^2, \]  
(18)
which can be viewed as a Schrödinger operator of a quantum particle in a potential well \( U(x) \) given by
\[ U(x) = -\frac{(\alpha + \epsilon)(\beta - \alpha)}{1 + \alpha\beta} z(x). \]  
(19)
Outside of the well (for \( |x| \to \infty \)), the potential approaches the asymptotic value
\[ U_{\text{out}} = -\frac{(\alpha + \epsilon)(\beta - \alpha)}{1 + \alpha\beta} z_{\text{out}}. \]  
(20)
A general solution for Eq. (16) is
\[ Q(x,t) = \sum_n C_n \exp(-\lambda_n t) Q_n(x), \]  
(21)
where \( \lambda_n \) and \( Q_n(x,t) \) are the eigenvalues and eigenfunctions of the operator \( \hat{H} \). In the limit \( t \to \infty \), the sum is dominated by the term with the largest negative eigenvalue \( \lambda_0 \) which corresponds to the deepest energy level in the quantum problem. The corresponding eigenfunction decays exponentially outside of the well, \( Q_0(x) \propto \exp(-\kappa |x|) \) with
\[ \kappa = \sqrt{\frac{1}{1 + \alpha\beta} - \frac{1}{\lambda_0 + U_{\text{out}}}}. \]  
(22)
Hence, we need to find both the lowest energy level of the potential \( U(x) \) and the respective eigenfunction.

We do not yet know the exact distribution \( z(x) \) for a stationary core. As an approximation (see also the next two sections), we take
\[ z(x) = \begin{cases} z_{\text{out}} & \text{for } |x| > R, \\ z_{\text{center}} & \text{for } |x| \leq R. \end{cases} \]  
(23)
The values \( z_{\text{center}} \) and \( z_{\text{out}} \) of the variable \( z \) inside and outside of the core and the half-width \( R \) of the stationary core will be determined later.

The eigenvalues and eigenfunctions of the Schrödinger operator for a quantum particle in the rectangular one-dimensional potential well can be found easily. The respective phase distribution may then be obtained by using Eq. (15). Here we omit this straightforward derivation and only give the final results. The phase distribution in the limit of large times outside of the core is
\[ \phi(x,t) = [\alpha + (\alpha + \epsilon)z_{\text{out}}] t + (\beta - \alpha) k^2 t - k |x| + \text{const.}, \]  
(24)
The wavenumber $k$ of the emitted waves is determined as a root of the equation
\[
\sqrt{k_{\text{max}}^2 - k^2 \tan \left[ \frac{\beta - a}{1 + a\beta} \sqrt{k_{\text{max}}^2 - k^2} \right]} = k,
\] (25)
where
\[
k_{\text{max}}^2 = \frac{\alpha + \epsilon}{\beta - a} (z_{\text{center}} - z_{\text{out}}).\] (26)

Note that the eigenvalue $\lambda_0$, $\kappa$, and the wavenumber $k$ are related through
\[
\kappa = \frac{(\beta - a)(1 + a\beta)^{-1} k}{1 + a\beta},
\]
\[
\lambda_0 = U_{\text{out}} - \frac{(\beta - a)^2 (1 + a\beta)^{-1} k^2}{1 + a\beta}.\]

### 3.2. Core dynamics

The core represents a region where the variable $z$ is increased with respect to its value $z_{\text{out}}$ in the outside region (Fig. 3). The boundaries of the core are formed by two interfaces (fronts) connecting the two states. For a steady pacemaker with a stationary core the fronts should not move. The dynamics of the variable $z$ is described by Eq. (10b).

If the terms $(\nabla \phi)^2$ and $\nabla^2 \phi$ are dropped in this equation, it takes the form of a standard equation describing front propagation in bistable media [42]. Such terms, however, cannot be neglected, because they lead to a significant renormalization of the front solutions. In the vicinity of a front, these terms can be determined as follows.

We assume that the core is close to its stationary solution so that the front velocities are very small. Therefore, the wave pattern adjusts to the instantaneous size of the core. The Hopf–Cole transformation (15) implies that the phase gradient $\nabla \phi$ and its derivative $\nabla^2 \phi$ are given by
\[
\nabla \phi = \frac{1 + a\beta}{(\beta - a)Q} \nabla Q, \quad (27a)
\]
\[
\nabla^2 \phi = \frac{1 + a\beta}{(\beta - a)Q^2} (Q \nabla^2 Q - (\nabla Q)^2). \quad (27b)
\]

As we have shown in the previous section, the distribution of $Q_0$ corresponding to the asymptotic wave pattern satisfies the equation
\[
-\lambda_0 Q_0 = \frac{(\alpha + \epsilon)(\beta - a) z(x)}{1 + a\beta} Q_0 + (1 + a\beta) \nabla^2 Q_0. \quad (28)
\]
It can be used to express $\nabla^2 Q_0$ as a function of $Q_0$ and $z$, namely
\[
\nabla^2 Q_0 = \kappa^2 Q_0 - \frac{(\alpha + \epsilon)(\beta - a) z(x)}{(1 + a\beta)} [z(x) - z_{\text{out}}] Q_0. \quad (29)
\]

On the other hand, we notice that the eigenfunction $Q_0$ of the rectangular potential well (22) and its derivative $\nabla Q_0$ should be continuous at the well boundary, i.e., at $|x| = R$. Therefore, there we have $\nabla Q_0 = -\kappa Q_0$. Thus, in the vicinity of the core boundary we obtain
\[
\nabla \phi = -k, \quad (30a)
\]
\[
\nabla^2 \phi = -\frac{\alpha + \epsilon}{1 + a\beta} [z(x) - z_{\text{out}}]. \quad (30b)
\]

Note that the second derivative of the phase is varying strongly within the core boundary. Substituting these two expressions into Eq. (10b), we obtain a closed equation that describes the slow motion of the fronts,
\[
\tau \zeta = (\sigma - \gamma + \eta k^2 - \nu u_{\text{out}}) z + (1 - \gamma + \nu a) z - \nu z^3 + \nu^2 \zeta, \quad (31)
\]
where $a = \beta(a + \epsilon)(1 + \alpha \beta)^{-1}$. Eq. (31) has the form of an equation for a bistable medium, but the coefficients are now renormalized and depend on the wavenumber $k$ of the emitted waves. The states of the medium on both sides of a front are given by the roots of the cubic equation

$$v z^3 - (1 - \gamma + \gamma a)z = \sigma - \gamma(1 + \alpha z_{\text{out}} - k^2).$$

(32)

The smallest root of this equation corresponds to the value $z_{\text{out}}(k)$ of the variable $z$ outside of the pacemaker core.

The motion of the front that connects both stable states is determined (cf. [42]) by the roots of the cubic equation (32). Since the quadratic term in this equation is absent, the sum of all three roots is zero and the front velocity $V$ is given by the middle root $z_{\text{middle}}(k)$ of Eq. (32)

$$V(k) = -\frac{3}{\tau} \sqrt{\frac{\nu}{2}} z_{\text{middle}}(k).$$

(33)

The velocity $V$ is a function of the wavenumber $k$ and can vanish at a certain critical wavenumber.

On the other hand, the value of the variable $z$ in center of the core can be determined from the following arguments:

Since the phase gradient $\nabla \phi$ vanishes at $x = 0$. Eq. (27b) reduces there to

$$\nabla^2 \phi \big|_{x=0} = (1 + \alpha \beta) \nabla^2 Q (\beta - \alpha).$$

(34)

Using Eq. (29), the phase curvature at $x = 0$ is obtained as

$$\nabla^2 \phi \big|_{x=0} = \frac{\beta - \alpha}{1 + \alpha \beta} k^2 - \frac{\alpha + \epsilon}{1 + \alpha \beta} (z_{\text{center}} - z_{\text{out}}).$$

(35)

where $k$ is the wavenumber of the pattern outside the core. Substituting this expression into Eq. (10b) and using that $\nabla \phi = 0$ in the center, we find that the following equation holds at $x = 0$

$$\tau \partial_t z = \sigma - \gamma - \gamma \left[ \frac{\beta(\beta - \alpha)}{1 + \alpha \beta} k^2 - a(z_{\text{center}} - z_{\text{out}}) \right] + (1 - \gamma)z - v z^3 + i^2 \nabla^2 z.$$

(36)

Hence, the value $z_{\text{center}}$ of the variable $z$ in the center of the pacemaker core is given by the largest root of the cubic equation

$$v z^3 - (1 - \gamma + \gamma a)z = \sigma - \gamma(1 + \alpha z_{\text{center}} - k^2).$$

(37)

### 3.3. Stationary pacemakers

Now we combine the results of our analysis in the two previous sections and determine the properties of self-organized pacemakers with stationary cores in a one-dimensional system. The core boundaries of such pacemakers should represent fronts whose velocity $V$ satisfies $V = 0$. According to Eq. (33), this implies that the middle root of Eq. (32) should be zero. Consequently, the wavenumber $k_0$ of the waves emitted by a stationary pacemaker should satisfy the equation

$$\sigma - \gamma + \gamma k_0^2 - \gamma a z_{\text{out}} = 0.$$  

(38)
If \( k = k_0 \), the other two roots of the cubic equation (32) can easily be found as

\[
\begin{align*}
\zeta_{\text{out}} &= \sqrt{1 - \gamma (1 - a) v}, \\
\zeta_{\text{in}} &= + \sqrt{1 - \gamma (1 - a) v},
\end{align*}
\]

(39a) (39b)

with \( a = \beta (\alpha + \epsilon) (1 + a \beta)^{-1} \). Since \( \zeta_{\text{out}} \) is now known, the wavenumber \( k_0 \) of the emitted waves can be determined from Eq. (38) as

\[
k_0 = \sqrt{1 - \sigma \gamma - a \sqrt{1 - \gamma (1 - a) v}}.
\]

(40)

Using Eq. (24), the frequency \( \omega_0 \) of generated waves is found as

\[
\omega_0 = \alpha - (\alpha + \epsilon) \sqrt{1 - \gamma (1 - a) v} + (\beta - \alpha) k_0^2.
\]

(41)

Finally, the half-width \( R_0 \) of the core of a stationary pacemaker is determined by Eq. (25) as

\[
R_0 = \frac{1 + a \beta}{(\beta - \alpha) \sqrt{k_{\text{max}}^2 - k_0^2}} \tan^{-1} \left( \frac{k_0}{\sqrt{k_{\text{max}}^2 - k_0^2}} \right),
\]

(42)

where \( k_{\text{max}}^2 = (\alpha + \epsilon) (\beta - \alpha)^{-1} (\zeta_{\text{center}}^2 - \zeta_{\text{out}}^2) \) and \( \zeta_{\text{center}} \) is given by the largest root of the cubic equation (37).

Analyzing the above results, we notice that the condition \( \gamma (1 - a) \leq 1 \) must be satisfied. Moreover, the wavenumber \( k_0 \) must not exceed \( k_{\text{max}} \), imposing additional restrictions on the model parameters, which are discussed further below.

In Fig. 4, we display the wavenumber \( k_0 \) of emitted waves, the half-width \( R_0 \) of the pacemaker core, and the pacemaker frequency \( \omega_0 \) as functions of the parameter \( \sigma \). These dependences are only shown within the interval of \( \sigma \) where the solutions actually exist. On the left border of the existence interval (at \( \sigma = \sigma_c \) as discussed below), the half-width \( R_0 \) of the core goes to zero, whereas the core size diverges on the right border (where \( k_0 \to k_{\text{max}} \) as discussed below).

The wavenumber \( k_0 \) of the emitted waves can be written as

\[
k_0 = \frac{\sigma_c - \sigma}{\gamma},
\]

(43)

Fig. 4. The wavenumber \( k_0 \), the half-width \( R_0 \), and the frequency \( \omega_0 \) of a stationary pacemaker as functions of the parameter \( \sigma \). The other parameters are \( a = 0.5, \beta = 1.5, \epsilon = 0.5, v = 40, \) and \( \gamma = -0.075 \).
where

\[ \sigma_c = \gamma - \gamma a_1 \frac{1 - \gamma(1 - \alpha)}{\nu} \].  \hspace{1cm} (44)

Note that pacemakers are found for \( \sigma < \sigma_c \) if \( \gamma > 0 \) and for \( \sigma > \sigma_c \) if \( \gamma < 0 \).

Near \( \sigma = \sigma_c \), the value \( z_{\text{center}} \) in the center of the pacemaker is approximately given by

\[ z_{\text{center}} \approx \sqrt{\frac{1 - \gamma(1 - \alpha)}{\nu}}. \]  \hspace{1cm} (45)

In the limit \( \sigma \to \sigma_c \), self-organized pacemakers have small cores, with a half-width \( R_0 \) given by

\[ R_0 \approx \frac{(1 + \alpha \beta) k_0}{(\beta - \alpha) k_{\text{max}}} \]. \hspace{1cm} (46)

Then, the frequency \( \omega_0 \) of emitted waves is

\[ \omega_0 \approx \omega_1 + (\beta - \alpha) k_{\text{max}}^2. \] \hspace{1cm} (47)

where \( \omega_1 \) is the frequency of the slow uniform oscillations, \( \omega_1 = \frac{\alpha + (\sigma + \epsilon) z_1}{\nu} \).

As seen from Eq. (42), the width of the pacemaker core diverges when \( k_0 \to k_{\text{max}} \). Rewriting Eq. (37) as

\[ v^2 z^2 - (1 - \gamma) z = \sigma - \gamma - \frac{\gamma(\beta - \alpha)}{1 + \alpha \beta} (k_0^2 - k_{\text{max}}^2), \] \hspace{1cm} (48)

it is evident that for \( k_0 \to k_{\text{max}} \), Eq. (37) reduces to Eq. (5), which determines the values of \( z \) in the uniform oscillatory states. Consequently, when \( k_0 \to k_{\text{max}} \), the value \( z_{\text{center}} \) of the variable \( z \) in the center of the pacemaker is close to \( z_1 \), and the frequency \( \omega_0 \) of the emitted waves satisfies

\[ \omega_0 \approx \omega_1 - (\beta - \alpha) (k_{\text{max}}^2 - k_0^2). \] \hspace{1cm} (49)

where \( \omega_1 = \alpha + (\sigma + \epsilon) z_1 \).

Therefore, we note that \( \omega_1 < \omega_0 < \omega_3 \). The frequency \( \omega_0 \) approaches the frequency of rapid uniform oscillations when the half-width \( R_0 \) diverges as \( R_0 \to \infty \) (and \( k_0 \to k_{\text{max}} \)). On the other hand, when the core is small, \( k_0 \) is also small and the frequency \( \omega_0 \) is close to \( \omega_1 \).

We now examine the assumptions which had to be made to derive this analytical solution and determine the parameter restrictions they impose.

As we have already pointed out, the amplitude equations (2a) and (2b) did not include the two terms proportional to \( u^2 z^2 \) and \( |w|^2 z^2 \) in the normal form (1a) and (1b) which is justified if \( \nu \gg 1 \). In addition, the phase gradients must be small (\( |\nabla \phi| \ll 1 \)) in order to use the phase dynamics approximation (9). Since \( \nabla \psi = -k \), this implies that the wavenumber \( k_0 \) of the emitted waves should be small, \( |k_0| \ll 1 \). As follows from the above analysis, \( |k_0| < |k_{\text{max}}| \) and \( k_{\text{max}} \propto \nu^{-1/4} \). Hence, \( \nu \approx 1/4 \) should hold.

In Eq. (10a) of the phase dynamics approximation, two terms proportional to \( V^2 z \) and \( \nabla z \) have been neglected, which is justified if \( |(\beta/2)V^2 z| \ll |(\alpha + \epsilon) z| \) and \( |\nabla \nabla \phi| \ll |(\beta - \alpha)|/2 \). If the parameters \( \alpha, \beta, \) and \( \epsilon \) are of order unity (as we always assume), this implies that \( |V^2 z| \ll |z| \) and \( \nabla z \ll \nabla \phi \). The gradients of the variable \( z \) are significant only within the boundaries of the pacemaker core which represent standing fronts of width \( l \). Hence, the condition \( |V^2 z| \ll |z| \) implies that \( l \ll 1 \). The other condition \( |\nabla z| \ll |\nabla \phi| \) is then automatically satisfied (providing that \( v^{1/4} \) is large).

On the other hand, our approximate analysis is valid only when the half-width \( R_0 \) and the wavelength \( 2\pi/k_0 \) are much larger than the front width \( l \). If we use the estimate (46) of the core size for small cores, we see that \( R_0 \approx v^{1/4} \).

Therefore, \( R_0 \gg l \) implies that \( l \ll v^{1/4} \), which also follows from the requirement that \( k_0 l \ll 1 \). Combining the
According to Eq. (46), pacemakers with vanishingly small core sizes $R_0$ are expected in the limit $\sigma \to \sigma_c$. However, we should remember that our approximate analysis is applicable only if the half-width $R_0$ is larger than the front width $l$. This means that $\sigma$ should not be too close to $\sigma_c$. In other words, $\sqrt{(\sigma_c - \sigma) T} / l \gg 1$ must be fulfilled.

The distribution of $\gamma$ inside the core was assumed to be flat, i.e., $\gamma = \gamma_{\text{center}}$ for $|x| \leq R$ and not only in the center ($x = 0$). To check this approximation, we compare $\gamma_{\text{center}}$ given by the largest root of the cubic equation (48), with the value $\gamma_{\text{m}}$ given by Eq. (39b), which describes the variable $\gamma$ immediately at the inner side of the front. This flat distribution is justified if $|\gamma_{\text{m}} - \gamma_{\text{center}}| \ll |\gamma_{\text{center}}|$. As follows from Eq. (45), this condition is indeed valid for pacemakers with small cores ($k_0 \ll k_{\text{m}}$). It becomes less accurate for pacemakers with larger cores, where $k_0$ approaches $k_{\text{m}}$. In this case, the results of the above analytical derivation provide only a qualitative description.

In Section 3.1, we have calculated the wavenumber of waves emitted by a pacemaker assuming a quasistationary core. This is justified when the relaxation time for radial perturbations of the core is large compared to the characteristic time needed by the wave pattern to adjust to the core size. We expect this to be true when the dynamics of the variable $\gamma$ is slow compared to the characteristic time scale for the evolution of the wave pattern, i.e., when $\tau \gg 1$. The stability of self-organized pacemakers is investigated by numerical simulations in the next section.

The derivation of the pacemaker solution has been performed above assuming that the conditions $\beta - \alpha > 0$ and $\alpha + \epsilon > 0$ are fulfilled, implying that the variable $\gamma$ is increased inside the core. Now we can discuss how the solution is modified in other possible cases.

If $\beta - \alpha > 0$ and $\alpha + \epsilon < 0$, the variable $\gamma$ is decreased inside the core. Then, the right-hand sides of the expressions (39a) and (39b) for $\gamma_{\text{m}}$ and $\gamma_{\text{m}}$ should be interchanged and Eqs. (40) and (41) for the wavenumber and the frequency must be modified accordingly. In this case, the value $\gamma_{\text{m}}$ is negative and given by the smallest root of Eq. (37).

When the dispersion is negative (i.e., $\beta - \alpha < 0$), the expression (42) for $R_0$ should be modified by replacing $\beta - \alpha$ by $\alpha - \beta$. If $\alpha + \epsilon > 0$, the variable $\gamma$ will be decreased inside the core. To obtain final results in this case, the expressions (39a) and (39b) for $\gamma_{\text{m}}$ and $\gamma_{\text{m}}$ must be interchanged and the equations for the wavenumber and frequency (Eqs. (40) and (41)) should be modified, respectively. Note that then $\gamma_{\text{m}}$ is given by the smallest root of Eq. (37). Finally, when $\alpha + \epsilon < 0$, the variable is increased inside the core. No modification of the results should be performed here besides the already mentioned replacement of $\beta - \alpha$ by $\alpha - \beta$.

3.4. Drift in a parameter gradient

Self-organized pacemakers are not pinned and their positions are determined by the initial conditions only. Such self-organized structures are therefore able to move through the medium if its properties are not uniform. Suppose that one of the parameters of the model, for example $\gamma$, is not constant, but varies in space as $\gamma = \gamma(x)$. The variation of this parameter is so smooth that its change on a distance equal to the half-width $R_0$ of the pacemaker core is small. In this case the velocity of the drift, induced by the parameter gradient, can be estimated using the constructed analytical solution and linear perturbation theory.

In the zeroth-order approximation with respect to the parameter gradient, we have a stationary pacemaker. For convenience, its spatial position will be chosen as the origin of the coordinate $x$, so in the vicinity of the center we have $\gamma(x) \approx \gamma_0 + \gamma x$. As already noted, the boundaries of the core represent two fronts whose motion is described by Eq. (31). The velocity $V$ of a front is determined by the middle root $\gamma_{\text{middle}}$ of the cubic polynomial (32), i.e. (cf. Eq. (33))

$$V = -\frac{1}{\tau} \frac{L}{\sqrt{2 \gamma_{\text{middle}}}}.$$

(51)
For the given model parameters, \( z_{\text{middle}} \) and \( V \) are functions of the wavenumber \( k \) of the emitted waves. For a stationary pacemaker, the core neither expands nor shrinks, and its boundaries must therefore represent standing fronts (\( V = 0 \)). This condition fixes the wavenumber \( k_0 \) of the waves emitted by a stationary pacemaker. If the model parameters are changing only little in space, the velocities of the two fronts, representing the left and the right boundaries of the core, will be slightly different and do not vanish. As a result, the pacemaker slowly drifts in space.

For sufficiently small gradients, we have

\[
V(\gamma(x)) \approx \frac{\partial V}{\partial \gamma} \bigg|_{\gamma_0} \frac{\partial \gamma}{\partial x} \bigg|_{x} = V'(\gamma_0) \chi x, \tag{52}
\]

where we have taken into account that \( V(\gamma_0) = 0 \) for a stationary pacemaker with wavenumber \( k_0 \). Hence, the velocities of the left and right fronts are

\[
V(\gamma(x = \pm R_0)) \approx \pm \chi R_0 V'(\gamma_0). \tag{53}
\]

We see that these two fronts would move with the same small absolute velocity, but in the opposite directions: one of them moves towards the center, whereas the other moves away from it. This means that the core is then drifting rigidly with the velocity

\[
V_D = -3 \frac{\gamma R_0}{\sqrt{\nu}} \frac{\partial V}{\partial \gamma} \bigg|_{\gamma_0} \frac{\partial \gamma}{\partial x} \bigg|_{x} \tag{54}
\]

When the middle root \( z_{\text{middle}} \) of Eq. (32) is small, it can be estimated by

\[
z_{\text{middle}} = \gamma \left(1 + a z_{\text{out}} - k^2 \right) - \sigma \frac{1 - \gamma + \gamma a}{1 - \gamma + \gamma a}. \tag{55}
\]

To determine the derivative of \( z_{\text{middle}} \) with respect to \( \gamma \) at \( \gamma = \gamma_0 \), we use that the coefficient \( a \) is independent of \( \gamma \) and that in zeroth-order approximation we can take the values of \( z_{\text{out}} \) and \( k \) given by Eqs. (39a) and (40) with \( \gamma = \gamma_0 \). The derivative \( z_{\text{middle}}'(\gamma_0) \) is then determined as

\[
z_{\text{middle}}'(\gamma_0) = \frac{1 + a z_{\text{out}} - k^2 + \sigma (a - 1)}{(1 - \gamma_0 + \gamma_0 a)^2}. \tag{56}
\]

Using the Eqs. (40), (51) and (54), we finally obtain

\[
V_D = -3 \frac{\gamma R_0}{\sqrt{\nu}} \frac{\sigma \chi R_0}{2 (1 - \gamma_0 + \gamma_0 a)} \tag{57}
\]

Thus, the drift velocity is proportional to the gradient \( \chi \) of the parameter \( \gamma \).

The above analysis refers to the case when the variable \( z \) is increased inside the core. The drift direction is reversed if the variable \( z \) is decreased inside the core region.

### 4. Numerical investigations

Eqs. (2a) and (2b) were integrated with an explicit Euler scheme where the Laplacian operator was discretized with a nearest neighbor approximation. The numerical accuracy was tested by repeating several simulations using a fourth-order explicit Runge–Kutta scheme and a stiff implicit Gear method. No significant differences were detected. No-flux boundary conditions were used. The aim of the simulations was to check the existence of self-organized pacemakers in one- and two-dimensional systems. Instabilities of these patterns and the effects of global inhibition were also numerically investigated.
Fig. 5. Development of a stable stationary pacemaker (a) and (b) and its asymptotic profile (c). Frame (a) shows the evolution of the amplitude of the real mode $z$ after an initial perturbation, whereas frame (b) displays the corresponding evolution of $\text{Re} A$. In frame (c) the spatial distributions of the variables $z$ (solid line), $\text{Re} A$ (dotted line) and $|A|$ (dashed line) are presented. The system size is $L = 100$ and the displayed time interval is $0 \leq t \leq 2000$ for $z$ and $0 \leq t \leq 200$ for $\text{Re} A$. The parameters are $\alpha = 0.5$, $\beta = 1.5$, $\epsilon = 0.5$, $l = 1$, $\tau = 1$, $\nu = 40$, $\gamma = -0.075$, and $\sigma = -0.06$. In the gray-scale plots, darker areas correspond to larger values of displayed variables.

4.1. Stable pacemakers and their drift

In a distributed system with birhythmic dynamics, the two states associated with uniform oscillations with different frequencies represent two attractors. Besides, stable self-organized pacemakers should also correspond to a certain attractor in the distributed dynamical system described by Eqs. (2a) and (2b). The coexistence of several attractors implies that the final state of the system may strongly depend on the initial conditions. When the conditions $\beta - \alpha > 0$ and $\alpha + \epsilon > 0$ are satisfied, a pacemaker can be created by starting from the uniform low-frequency oscillatory state and applying a sufficiently strong local perturbation of the variable $z$ (increasing the variable $z$ from $z_1$ close to $z_3$ inside a region with a certain half-width $R_0$). Note that the attraction basin for the pacemakers with small cores may be very small, and therefore only perturbations with a half-width close to the stationary value $R_0$ of the pacemaker core will converge to the pacemaker solution.

Fig. 5 shows the development of a pacemaker in the one-dimensional system from the initial state, consisting of a sufficiently large perturbation to its asymptotic state. In the space–time diagrams (a) and (b) time runs along the horizontal axis. In the gray-scale representation used in this figure, darker regions correspond to larger values of the displayed variables. We see that the perturbation started to expand and to emit waves. The expansion speed is decreasing until the expansion finally terminated and a stationary core, emitting waves with a constant wavenumber, has been formed. Snapshots of the spatial distributions of the real part $\text{Re} A$ of the complex oscillatory amplitude $A$, the modulus $|A|$, and $z$ for a stable stationary pacemaker are shown in Fig. 5(c).

If the conditions $\beta - \alpha > 0$ and $\alpha + \epsilon < 0$ are satisfied instead, the variable $z$ is decreased inside the core and the initial perturbation must therefore be modified accordingly. This case was realized in the simulation for the two-dimensional medium, shown in Fig. 6. The gray-scale representation for the variable $z$ is inverted, so that the smaller values of $z$ correspond to the darker areas here.

In our simulations, the parameters of the model did not satisfy the conditions (50) and therefore only the existence of stable self-organized pacemakers in one- and two-dimensional systems was confirmed without quantitatively verifying the constructed analytical solutions.

It was shown in Section 3.4 that the pacemakers should drift when spatial parameter gradients are present. Fig. 7 shows a simulation confirming this effect. The simulation was initiated with a stable stationary pacemaker. A constant gradient in the parameter $\gamma$ was then introduced and the pacemaker started to drift through the medium.
Fig. 6. A two-dimensional stable stationary pacemaker. The spatial distributions of variables \( z \) (a) and \( \text{Re} A \) (b) are displayed. The system size is \( L_x = L_y = 120 \); the parameters are \( \alpha = -1.0, \beta = -0.5, \epsilon = 0.3, l = 1, r = 3, \nu = 40, \gamma = 0.20, \) and \( \sigma = 0.16 \). The gray-scale representation for variable \( z \) is inverted here (smaller values of \( z \) correspond to the darker regions).

in the direction of increasing \( \gamma \). When the gradient was removed, the motion of the pacemaker terminated and a stationary pacemaker was recovered at a new location, as seen from Fig. 7(a). Emission and propagation of waves persisted during the drift (Fig. 7(b)). Note that the drift velocity is much smaller than the phase velocity of the waves.

4.2. Global inhibition

In oscillatory media with positive dispersion, wave sources with a higher frequency suppress all less rapid sources [2]. Therefore, the competition between usual pacemakers in heterogeneous media is always won by the most rapid pacemaker which suppresses all others. If two pacemakers have exactly the same frequency, they should coexist indefinitely. Since the frequencies of all self-organized pacemakers in the same birhythmic medium are equal (they are uniquely determined by the parameters of the system), this suggests that such pacemakers must always coexist. Our numerical simulations reveal, however, that the actual behavior is more complex.

To investigate interactions between pacemakers, a series of two-dimensional simulations was performed. In these simulations, a pacemaker was first created. After its wave pattern had spread over the entire medium and the core had reached its stationary radius, another pacemaker was initiated by applying a strong local perturbation of \( z \) at some distance from the center of the original pacemaker. If the already existing pacemaker were absent, the new perturbation would have evolved into a stationary pacemaker. A typical actual evolution, observed for sufficiently

Fig. 7. Drift of a pacemaker. The spatial gradient of the parameter \( \gamma \) is applied inside the time interval indicated by the vertical dashed lines in frame (a), which shows the evolution of \( z \) in the time interval \( 0 \leq t \leq 2 \times 10^5 \). The pacemaker is drifting in the direction of increasing \( \gamma \). The parameters are \( \alpha = 1.38, \beta = 2.3, \epsilon = -3.18, l = 0.8, r = 2, \nu = 83, \gamma_0 = 5.59 \times 10^{-4}, \gamma = 1.68 \times 10^{-4}, \) and \( \sigma = 3.4 \times 10^{-4} \). Frame (b) displays the drifting wave pattern within a narrow time interval \( \Delta t = 500 \) during the drift, marked by the dotted vertical line in frame (a). The gray-scale representation for \( z \) is inverted.
Fig. 8. Interaction of a stationary pacemaker with another perturbation, leading to fusion of the respective cores and global inhibition of other pacemakers. Snapshots of $z$ are shown in frames (a)–(d), snapshots of Re $A$ in frames (e)–(h). The displayed time moments are $t = 100$ (a) and (e), $t = 500$ (b) and (f), $t = 700$ (c) and (g), and $t = 1500$ (d) and (h). In frame (i) a space–time diagram of $z$ for a cross-section through the centers of the two cores is shown for the time interval $0 \leq t \leq 2000$. The parameters are $L_x = 60$, $L_y = 120$, $a = -1$, $b = -0.5$, $\epsilon = 0.5$, $\beta = 1$, $\tau = 9$, $\nu = 40$, $\gamma = 0.2$, and $\sigma = 0.16$. The gray-scale representation for $z$ is inverted.

Large separations, is shown in Fig. 8. The perturbation created a core (a) which started to grow (b). However, at the same time, the core slowly drifted towards the (large) pacemaker which remained immobile (c). Eventually, the two cores met and fused. Immediately after the fusion, the resulting pacemaker had a core which was larger than the one of a stationary pacemaker. Subsequently, the core shrunk back to the size corresponding to a stable stationary pacemaker (d). A diagram showing the evolution of the spatial distribution of $z$ along the central vertical cross-section of the medium is presented in Fig. 8(i). Additionally, in Fig. 8(e)–(h) we display four snapshots of the wave patterns during this evolution. We see that the second core was not able to emit waves independently (and should therefore not be called a pacemaker), but only modified the wave pattern generated by the (large) pacemaker.

For smaller spatial separations or for weaker perturbations, the observed evolution is different: Although the second perturbation would be still large enough to create a pacemaker in the absence of another core, it was suppressed immediately and did not give rise to a growing core. In Fig. 9, we show how the size of a small perturbation and the distance from a stationary pacemaker determine whether fusion (Fig. 8) or immediate suppression take place. Note that for very small initial separations the cores interacted directly and fusion was again observed.
This behavior can be characterized as *global inhibition*: If a pacemaker has developed and its wave pattern has covered the whole medium, the formation of further pacemakers becomes impossible in the entire medium. Perturbations, which otherwise would have been sufficient to create a pacemaker, were either damped immediately or gave rise to localized perturbations which drifted towards the dominant pacemaker and finally fused with it.

The situation is different if the second perturbation was applied to a region which has not yet been reached by the spreading wave pattern of the first pacemaker. In this case, the second pacemaker stabilized and developed its own pattern of emitted waves, which eventually led to the formation of a state where the coexistence of the two pacemakers is observed. In general, this occurs when the time interval between two subsequent perturbations (well separated in space) is relatively short.

### 4.3. Instabilities of self-organized pacemakers

The stability of pacemaker cores is affected by the ratio \( \tau \) of the two characteristic times in the model equations (2a) and (2b). We have seen in Section 4.1 that the cores are stable with respect to radial perturbations if the variable \( z \) is slow compared to the complex amplitude \( A \) of the oscillatory mode, i.e., if the ratio \( \tau \) is relatively large. When \( \tau = 1 \) (Fig. 5), the transient leading to the stable pacemaker was monotonous. However, when \( \tau \) was decreased, we observed that the transients ceased to be monotonous. The half-width \( R \) of the core did not grow until its stationary value \( R_0 \) was reached, but first got larger and then decreased to \( R_0 \). When \( \tau \) was further decreased, damped oscillations of the core were observed. At still smaller \( \tau \), these oscillations became sustained and pacemakers with breathing cores are found, as shown in Fig. 10(a) and (b). This breathing behavior resembles breathing known for localized spot patterns in activator–inhibitor systems [43]. Swinging pacemakers have also been observed in our simulations (Fig. 10(c) and (d)). In this case, the half-width of the core remained approximately constant whereas the position of the pacemaker oscillated in space. A decrease in \( \tau \) for breathing pacemakers resulted in an increasing amplitude of core oscillations which became strongly unharmonic. For even lower \( \tau \), the core oscillations were so large that they either led to a collapse and the disappearance of the pacemaker or resulted in a rapid expansion of the core until the whole medium was transformed into the other uniform oscillatory state. Which of the two possibilities was realized depended strongly on the initial conditions. Breathing pacemakers were also found in two-dimensional simulations.

Another mechanism, leading to destabilization of a pacemaker and expansion of its core, involves phase slips. This is related to the fact that oscillatory media cannot support propagation of waves if their wavenumber exceeds a certain threshold. The amplitude \( |A| \) of plane waves in the CGLE depends on the wavenumber \( k \) as \( |A| \propto \sqrt{1-k^2} \).
Fig. 10. Breathing (a) and (b) and swinging (c) and (d) pacemakers. The displayed coordinate and time intervals are $\Delta x = 100$, $\Delta t = 250$. In frames (a) and (b) $\beta = 3.0$ and in frames (c) and (d) $\beta = 2.65$. The other parameters are $L = 200$, $\alpha = 1.38$, $\epsilon = -3.18$, $I = 0.8$, $r = 0.001$, $\nu = 83$, $\gamma = 5.59 \times 10^{-4}$, $\sigma = 3.4 \times 10^{-7}$. The gray-scale representation for $z$ is inverted.

and propagation of waves with $k > 1$ is therefore impossible. Moreover, even before this propagation failure boundary is reached, the waves become unstable with respect to the modulational Eckhaus instability found at $k = k_{EI} < 1$ (see [44]). The subcritical variant of this instability is associated with phase slips, which occur when the oscillation amplitude locally drops down to zero at a certain time moment. Then, the oscillations are locally absent and their phase is not defined at the respective point. There, the phase can jump by $2\pi$ which corresponds to the disappearance of one wave (i.e., of one oscillation period) in a wave train.

For pacemakers in heterogeneous oscillatory media, where the core represents a fixed region with an increased local oscillation frequency, phase slips develop if the frequency difference of local oscillations inside and outside the core is sufficiently large [38] (see also [41]). Then, the wavenumber of the emitted waves is also large and may overcome the threshold $k_{EI}$, which for the CGLE is given by $k_{EI}^2 = (1 + \alpha \beta)/(3 + \alpha \beta + 2\sigma^2)$. In this case, the phase slips are repeatedly occurring at some distance from the core and only some waves initiated inside the core are able to pass through the zone, where phase slips take place, and propagate outside.

For self-organized pacemakers in birhythmic media, a similar effect can be expected. The change of the oscillation frequency inside the core, however, is no longer determined by a fixed inhomogeneity, but by a local increase of the dynamical variable $z$. When phase slips are generated, this may have an effect on $z$ and on the dynamics of the core itself.

Our numerical simulations show that generation of phase slips leads to the destabilization of the core. An example of this instability is displayed in Fig. 11. We see that oscillations inside the core are more rapid than in the periphery and that the zone where phase slips occur lies very close to the core boundary. Only some of these internal oscillations are able to emit waves (Fig. 11(b)). The space–time plot of the modulus $|A|$ of the oscillation amplitude (Fig. 11(c)) clearly shows the periodic development of amplitude defects where $|A| = 0$. This process is accompanied by the gradual growth of the core (Fig. 11(a)) which proceeds until the whole medium is dominated by the uniform state with the highest oscillation frequency.

Note that the width of the expanding core is weakly modulated: Just before the occurrence of a phase slip, the local wavenumber increases strongly, leading to a slight retreat of the two core fronts. After the appearance of the phase slip, the local wavenumber decreases, implying that the core fronts move outwards, which gives rise to a core expansion. The expansion prevails over the contraction, and the net effect is a slow growth of the pacemaker core.
Fig. 11. Destabilization of a pacemaker by phase slips. The displayed space and time intervals are $\Delta L = 100$ and $\Delta t = 625$. The parameters are $L = 100$, $\alpha = 1.36$, $\beta = 2.1$, $s = -3.18$, $\epsilon = 0.025$, $r = 85$, $y = 5.59 \times 10^{-4}$, and $\sigma = 3.4 \times 10^{-4}$. The gray-scale representation for $z$ is inverted.

5. Discussion

The principal result of our study is that self-organized pacemakers represent a generic wave pattern in oscillatory media near the soft onset of birhythmicity described by a supercritical pitchfork–Hopf bifurcation. The physical mechanism responsible for the stabilization of pacemakers in the considered system involves a long-range negative feedback, similar to the one necessary for the formation of stable localized spots in reaction–diffusion models with fast inhibitor diffusion. An infinite-range inhibition, however, is caused here not by diffusion, but by the non-damped propagation of waves emitted from the core region.

We have analytically constructed a solution for self-organized pacemakers in the vicinity of a pitchfork–Hopf bifurcation. The frequency and the wavenumber of the waves emitted by a stationary self-organized pacemaker, and the size of its core, have been analytically estimated. The drift of self-organized pacemakers due to spatial parameter gradients has been predicted and the drift velocity has been estimated. The effect of the pacemaker drift in systems with spatial parameter gradients provides a convenient experimental method to identify self-organized pacemakers and distinguish them from other target patterns caused by local heterogeneities in the medium. Our numerical investigations have confirmed the existence of self-organized pacemakers and have shown that they are stable for a range of the model parameters. We were also able to see the drift of pacemakers in non-uniform media and considered interactions between the pacemakers. Instabilities of pacemakers, leading to their breathing, swinging, and expansion, have been found numerically.

Since our analysis is based on general amplitude equations, it is valid for any reaction–diffusion system near the soft onset of birhythmicity with small-amplitude limit cycles. As in the case of a Hopf and Turing–Hopf bifurcation, we expect that the results of such an analysis, based on amplitude equations would remain (at least, qualitatively) applicable even at larger distances from the bifurcation point where the amplitude equations are no longer strictly applicable.

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References


