Controlling turbulence in the complex Ginzburg–Landau equation II. Two-dimensional systems

D. Battogtokh\textsuperscript{a,1}, A. Preusser\textsuperscript{b}, A. Mikhailov\textsuperscript{a,*}

\textsuperscript{a} Fritz-Haber-Institut der Max-Planck-Gesellschaft, Faradayweg 4-6, 14195 Berlin (Dahlem), Germany
\textsuperscript{b} Gemeinsames Rechenzentrum der Berliner Max-Planck-Institute, Faradayweg 4-6, 14195 Berlin (Dahlem), Germany

Received 4 October 1996; revised 12 December 1997; accepted 6 March 1997
Communicated by Y. Kuramoto

Abstract

Turbulence in oscillatory distributed systems can be controlled by introducing a delayed global feedback and adjusting the feedback intensity and the delay time. We investigate influence of global feedbacks on turbulence in two-dimensional systems described by the complex Ginzburg–Landau equation. Inside a synchronization window, application of such feedbacks leads to destruction of phase flips and spiral waves, appearance of breathing and stationary cellular structures or stripes, and development of localized turbulent bubbles on the background of uniform oscillations.

1. Introduction

The complex Ginzburg–Landau equation (CGLE) represents a normal form of a distributed dynamical system in the vicinity of a supercritical Hopf bifurcation [1,2]. Though individual elements of a medium have in this case simple regular dynamics, i.e. perform harmonical limit-cycle oscillations, local diffusional coupling between them can produce turbulence [3]. Because of its generality, CGLE has been extensively studied both in its regular and turbulent parameter domains [4–14]. Using this model, such concepts as phase, amplitude, and defect turbulence have been introduced [3,8].

Under a delayed global feedback, elements of a distributed oscillatory system collectively generate a control signal that is applied back to each of them after adding a certain delay. Depending on the phase shift of the control signal, the feedback can be either negative or positive, and thus it may alternatively suppress spatiotemporal chaos or destabilize uniform oscillations. The principal role of delays is that they modify phase shifts between the control signal and the oscillating pattern. By changing the delay, the phase shift can be directly varied.

* Corresponding author.
1 Present address: Department of Physics, Kyoto University, Kyoto 606, Japan. On leave from: Physics and Technology Institute, Mongolian Academy of Sciences, Ulanbator, Mongolia.
The dimensionality of a system is known to play an essential role in the phenomena related to fluctuations and spatiotemporal chaos near the bifurcation points. For example, statistical fluctuations destroy long-range order in one-dimensional equilibrium systems [15]. On the other hand, resonant interactions between triplets of plane wave modes, leading to the appearance of hexagonal structures under the Turing instability [16,17], are first possible only in two dimensions.

In the previous publication [18], hereafter referred as Part I, we have shown that turbulence in one-dimensional systems, described by CGLE, can be controlled by introducing a delayed global feedback and adjusting the feedback intensity and the delay time. The aim of the present paper is to provide classification of typical spatiotemporal regimes found under application of global feedbacks in oscillatory two-dimensional reaction–diffusion systems near the onset of oscillations.

We analyze principal scenarios leading to suppression of turbulence by an increasing global feedback and investigate patterns that appear when the feedback intensity is fixed slightly under the synchronization threshold. These new patterns include stationary and breathing cellular structures or stripes, turbulent bubbles on the background of uniform oscillations, and strings representing extended amplitude defects. Destruction of spiral waves and propagating phase flips by sufficiently strong global feedbacks is also described.

Our research is motivated by current experiments with catalytic surface chemical reactions and the Belousov–Zhabotinskii reaction. We have discussed in Part 1 how global delayed feedbacks can be artificially introduced when oscillatory chemical reactions are experimentally investigated. However, such feedbacks may also represent an intrinsic feature of a system. Particularly, global coupling is typical for low-pressure experiments with surface chemical reactions where it is due to strong mixing of reactive components in the gas phase, while local coupling is provided by surface diffusion [19]. Effects of global coupling have been observed in oscillatory surface reactions and discussed in the framework of CGLE with instantaneous global feedback [20–25]. It should also be noted that global feedback can be realized in distributed oscillatory optical systems.

Realistic oscillatory systems studied in experiments are seldom exactly described by CGLE, since this equation holds only in a close neighborhood of the Hopf bifurcation. However, predictions based on this model are widely used for qualitative interpretation of the experimental data even when this approximation does not directly apply. We believe that a similar approach can be fruitful when effects of a global feedback in oscillatory systems are considered. In accordance with these objectives, we focus our attention on the qualitative aspects of phenomena related to global feedbacks in CGLE. The core of this study consists of systematic numerical simulations, performed by monitoring responses of the considered system to variations of its parameters and feedback properties. The framework for our analysis is provided by previous studies of CGLE in absence of global feedback and, especially, by the recently constructed phase diagram for turbulence in this equation [13].

When pattern formation in nonlinear reaction–diffusion systems is considered, much information is contained in the data showing how one kind of patterns is transformed into a different kind as a control parameter is gradually varied. We usually begin with ramping simulations, i.e. when one of the system’s parameters is slowly increased with time, and when a pattern with interesting properties is found repeat the simulation at fixed parameters.

The CGLE with a global feedback is introduced in Section 2. In Section 3 we perform a stability analysis of uniform oscillations that determines boundaries of the synchronization window and properties of first unstable modes. Phase flips in CGLE with a global feedback are considered in Section 4. We derive an equation that determines properties of these structures at small feedback intensities. Numerical simulations show that, at larger feedbacks, phase flips become unstable and transform into uniform oscillation through a phase singularity. In two-dimensional systems, spontaneous break-up of a curved phase-flip wave has been observed. It is preceded by development of an extended amplitude defect which we call a string. The string breaks and topological defects appear at its open ends.

Destruction of spiral waves by global feedbacks is investigated in Section 5. We find that strong global feedbacks localize rotating spiral waves inside roughly circular regions. Such spiral waves are formed by phase-flip waves
with topological defects sitting in the center and at the outer end point. The regions populated by spiral waves shrink and uniform oscillations are eventually established.

In Section 6 we consider suppression of developed defect turbulence by increasing global feedback and present three scenarios leading to establishment of uniform oscillations in the system.

In the following sections we analyze properties of spatiotemporal patterns found at a fixed feedback intensity below the synchronization threshold. Cellular structures are investigated in Section 7. We show that such structures are produced by a process of ordering of the cells that form the state of phase turbulence in CGLE. At smaller feedback intensities, periodic regular breathing of cells develops. Regular stripe structures have also been observed in a narrow parameter interval.

Intermittent turbulence and turbulent bubbles on the background of uniform oscillations are studied in Section 8. The bubbles are formed by a cluster of active cells. In the center of such a cell, string loops are repeatedly appearing. These circular amplitude defects spread and break up, similar to the process observed in break-up of phase-flip waves in Section 4. We also analyze a possible role of small local inhomogeneities and show that they pin turbulent bubbles in the medium. The paper ends with conclusions and discussion of obtained results.

2. Global feedback in the complex Ginzburg–Landau equation

Using dimensionless time, coordinate and amplitude variables, the Ginzburg–Landau equation for local complex oscillation amplitudes $A(r, t)$ in the presence of an additive force $F(t)$ is written in the form

$$
\dot{A} = (1 - i\omega)A - (1 + i\beta)|A|^2A + (1 + i\epsilon)\nabla^2A + F(t).$$

In this equation $\beta$ characterizes the nonlinear frequency shift of individual oscillators and $\epsilon$ controls dispersion of traveling waves.

The linear oscillation frequency of individual oscillators in Eq. (1) is $\omega$. The derivation of CGLE as a normal form near a supercritical Hopf bifurcation (see [3]) is based on the assumption that the considered system is close to the bifurcation point and therefore the oscillation frequency is large as compared to the characteristic increment of growth of first unstable mode. Because the increment is made equal to unity in Eq. (1) by rescaling of time, this assumption is equivalent to the condition $\omega \gg 1$.

When global feedback is present, force $F(t)$ is constructed by taking the global average

$$
\overline{A}(t) = \frac{1}{S} \int_S A(r, t) \, dr
$$

(2)
of local complex oscillation amplitudes at a delayed time moment $t - \tau$ (here $S$ is the total area of the system) and multiplying this by a complex proportionality coefficient, i.e. as

$$
F(t) = \mu e^{i\chi_0} \overline{A}(t - \tau).$$

(3)

Hence, $\mu$ specifies the intensity of global feedback, $\chi_0$ characterizes the phase shift between the delayed average oscillation amplitude and the (complex) control signal, and $\tau$ represents the delay time.

It is convenient to introduce slowly varying complex oscillation amplitudes as

$$
\eta(r, t) = A(r, t) \exp(i\omega t).
$$

(4)

With this transformation, Eq. (1) takes the form

$$
\dot{\eta} = \eta - (1 + i\beta)|\eta|^2\eta + (1 + i\epsilon)\nabla^2\eta + f(t),
$$

(5)
where
\[ f(t) = \mu \exp[i(x_0 + \omega \tau)] \bar{\eta}(t - \tau) \] (6)
and
\[ \bar{\eta}(t) = \frac{1}{S} \int_{\mathcal{S}} \eta(\mathbf{r}, t) \, d\mathbf{r}. \] (7)

Characteristic evolution times of thus defined slow complex oscillation amplitudes \( \eta(\mathbf{r}, t) \) are of order unity. Therefore, when only short delays (\( \tau \ll 1 \)) are considered, as assumed throughout this paper, a delay between the slowly varying complex control signal \( f(t) \) and the slowly varying global oscillation amplitude \( \bar{\eta} \) in Eq. (6) can already be neglected.

Once this approximation is made, the only (but very important) remaining effect of delays is that they control the effective phase shift,
\[ \chi = x_0 + \omega \tau, \] (8)
between the global oscillation amplitude and the control signal. Note that since \( \omega \gg 1 \) the renormalization of the phase shift can be significant even for short delays, less than an oscillation period.

When such conditions are satisfied, the system is approximately described by the equation
\[ \dot{\eta} = \eta - (1 + i\beta)|\eta|^2\eta + (1 + ie)\nabla^2 \eta + \mu e^{i\chi} \bar{\eta}, \] (9)
where \( \bar{\eta} \) is given by the spatial average (7) of the complex oscillation amplitude. This equation has earlier been formulated to model effects of global coupling in oscillatory surface reactions \([21-25]\). Below we use Eq. (9) as a basic model for our analytical and numerical investigations.

In the absence of global coupling (\( \mu = 0 \)) Eq. (9) reduces to CGLE. When the boundary \( 1 + e\beta = 0 \) of the Benjamin–Feir (BF) instability is crossed, the uniform state in CGLE becomes unstable and phase turbulence develops. In this chaotic regime the medium is filled with an irregular set of cells whose boundaries are formed by ‘shocks’ where the local oscillation amplitude is increased. If, after crossing the BF boundary, we pass the phase turbulence state and move further into the BF-unstable region, amplitude turbulence is found in CGLE. This turbulence is characterized by strong variations of both the phase and the modulus of the complex oscillation amplitude. Transition to amplitude turbulence proceeds through formation of a small seed on the background of phase turbulence. This seed grows until the new regime occupies the entire medium. The transition is characterized by a hysteresis: starting from this state and moving back towards the BF boundary, amplitude turbulence may be found in the parameter regions where phase turbulence has been present under motion in the opposite direction.

For the amplitude turbulence, different states are distinguished. Closer to the BF boundary, frozen states are observed \([13]\). In such states the medium is filled with a few spiral waves. These waves are steadily rotating and their centers retain positions they have reached at the end of the initial transient process. The areas occupied by individual spirals are separated by shock lines, along which emitted waves collide and annihilate. These shocks are also approximately stationary. If the spatial distribution of the modulus \( \rho \) is plotted for a frozen state, point-like defects, where the modulus reaches zero, are clearly seen. They correspond to centers of spiral waves and are surrounded by flat regions where the modulus is constant. The state of defect turbulence is generally found farther away from the BF boundary. In this state, multiple topological defects, representing singularities of the phase field and therefore zeros of the oscillation amplitude, are observed. These defects randomly move through the medium.

Below in this paper we always consider (except for Section 4) the system described by Eq. (9) when the condition \( 1 + e\beta < 0 \) is satisfied, so that uniform oscillations are not stable without the global feedback.
The global feedback modifies the frequency and the amplitude of uniform oscillations. Substituting \( \eta(t) = \rho_0 e^{-i\Omega t} \) into Eq. (9), we find that in the presence of global feedback the oscillation frequency is shifted from \( \omega_0 \) by

\[
\Omega = -\mu (\sin \chi - \beta \cos \chi),
\]

and the modulus of the complex amplitude for uniform oscillations is

\[
\rho_0 = (1 + \mu \cos \chi)^{1/2}.
\]

When intensity \( \mu \) of global feedback is low, these corrections are small.

It is convenient to introduce the local modulus \( \rho \) of the complex oscillation amplitude and the local oscillation phase \( \phi \) as

\[
\eta(r, t) = \rho(r, t) \exp(-i\Omega t - i\phi(r, t)).
\]

The average (7) of the complex oscillation amplitude can also be written as

\[
\bar{\eta}(t) = R(t)e^{i\Omega t - i\Psi(t)}.
\]

Here \( \Psi(t) \) represents the global oscillation phase. The variable \( R(t) \) is called below the synchronization parameter. Indeed, it is given by the equation

\[
R(t) = \frac{1}{S} \left| \int \eta(r, t) \, dr \right|.
\]

Therefore, it vanishes when oscillations in different parts of the medium are not correlated and approaches its maximal value \( \rho_0 \) for the uniform oscillations.

To study nonlinear processes in the system, numerical simulations were performed. We employed an explicit integration method with constant time and coordinate steps \( dt \leq 0.01 \) and \( dx = 0.5 \) on a square grid of \( 200 \times 200 \) elements. The integration process was numerically stable and the results did not change when smaller integration steps have been chosen. The medium of size \( L = 100 \) was usually sufficiently large to contain many elementary structures. Then the results of a simulation did not significantly depend on the size of the system. However, we have found also single large patterns (such as e.g. phase domains) that occupied a considerable part of the medium. In this case a sensitive dependence on the size of the medium is expected and the results may be different when larger systems are considered.

No-flux boundary conditions were used, though some simulations were repeated with periodic boundary conditions. As the initial condition, the unstable uniform state \( \eta = 0 \) with weak random perturbations has usually been taken.

A detailed phase diagram for turbulence in the two-dimensional CGLE has recently been constructed [13]. As a test, we have checked that our simulations correctly reproduce all turbulent states reported in this publication.

Since we wanted to examine how the system responds to global feedback in different parameters regions, simulations were often first performed under ramping conditions. We started from the system in absence of global feedback \( (\mu = 0) \) and then slowly increased its intensity \( \mu \) at a constant speed, until uniform oscillations appeared (if we were inside the synchronization window of phase shifts). The detailed investigations of patterns existing below the synchronization threshold were then carried out at a fixed feedback intensity.

The data of each simulation have been used to generate computer animations consisting from 200 to 2000 single frames. In this paper, only selected frames from such animations are presented. Short videos, illustrating basic synchronization scenarios, are available via Internet [26].
For visualization of computed patterns, we employed different variables, i.e. the real part \( u = \text{Re} \eta \) of the complex oscillation amplitude \( \eta \), its modulus \( \rho \) and the phase \( \phi \). The spatial distributions of these variables at subsequent time moments have been plotted using different color codes or gray-scale coding. The spatial distribution of the amplitude modulus \( \rho \) is always slowly changing with time, whereas the variables \( u \) and \( \phi \) undergo large variation within a single oscillation period. To eliminate, as much as possible, such obvious rapid variation, we used coordinate frames which rotated at a frequency which was only slightly less than the actual frequency of oscillations. Therefore, the plotted distributions correspond rather to the variables \( \tilde{u} = \text{Re} (\eta e^{i\Omega_0 t}) \) and \( \tilde{\phi} = \phi + \Omega_0 t \) where \( \Omega_0 \) is the rotation frequency of the coordinate frame. Note that our basic equation (9) is already written in the coordinate frame that rotates at the frequency \( \Omega \) of uniform oscillations. However, when the oscillations in the medium are not uniform, their frequency differs, due to the effects of global feedback, from that of the uniform oscillations. In practice, we adjusted the rotation frequency \( \Omega_0 \) used in the visualization for each sequence of frames after it has been computed.

The phase was calculated as \( \phi = \text{Arg} \eta \). Hence, the phase varies inside the interval from 0 to \( 2\pi \). When it reaches its maximal possible value \( 2\pi \), the phase abruptly falls to the minimal bound 0. As a result, visualizations employing continuous gray-scale coding in the interval from 0 to \( 2\pi \) for the phase variable show sharp phase front that represent lines of the constant phase \( \phi = 0 \).

Note that the phase variable is often better suited for visualization of oscillations than real or imaginary parts of the complex oscillation amplitude. Since \( \text{Im} \eta = \rho \sin \phi \), the phases \( \phi \) and \( \pi - \phi \) yield the same value of the imaginary part \( \text{Im} \eta \). Therefore, if only the imaginary part of the oscillation amplitude is plotted, it is impossible to see whether a full phase rotation or only a variation of the phase inside the interval from 0 to \( \pi \) takes place along a certain direction.

To monitor the degree of synchronicity of oscillations, we followed the time dependence of the synchronization parameter \( R(t) \), defined by Eq. (14) and representing the modulus of the averaged oscillation amplitude.

3. Stability of uniform oscillations

To get insight into the expected behavior of the system in different parameter regions, we start with the linear stability analysis of uniform oscillations in the presence of global coupling. This analysis is made in close analogy with the respective stability investigation of uniform oscillations in CGLE under periodic external forcing [27,28]. Assuming periodic boundary conditions, complex oscillation amplitudes can be decomposed into a superposition of plane waves:

\[
\eta(r, t) = \sum_k \eta_k(t) e^{ikr}.
\] (15)

The equations determining evolutions of amplitudes \( \eta_k \) are obtained by performing a Fourier transformation of Eq. (9), which yields

\[
\dot{\eta}_k = (1 + \mu e^{i\chi} \Delta(k) - k^2 - i\beta k^2)\eta_k - (1 + i\beta) \sum_{k_1,k_2,k_3} \eta^*_{k_1} \eta_{k_2} \eta_{k_3} \Delta(k + k_1 - k_2 - k_3). \] (16)

The mode with \( k = 0 \) corresponds to uniform oscillations. As follows from (16), its amplitude \( \eta_0 \) obeys the equation

\[
\dot{\eta}_0 = (1 + \mu e^{i\chi})\eta_0 - (1 + i\beta)|\eta_0|^2\eta_0 - 2(1 + i\beta) \sum_{k \neq 0} |\eta_k|^2 \eta_0
\]

\[
- (1 + i\beta) \sum_{k \neq 0} \eta_k \eta_0 - (1 + i\beta) \sum_{k_1,k_2,k_3 \neq 0} \eta^*_{k_1} \eta_{k_2} \eta_{k_3} \Delta(k_1 - k_2 - k_3). \] (17)
When the stability of uniform oscillations is investigated, the plane wave modes with $k \neq 0$ represent small perturbations. Examining Eq. (17), it can be seen that in the linear approximation the dynamics of uniform oscillations is not influenced by such perturbations.

This means that in this approximation the plane-wave modes are decoupled from uniform oscillations and, when their evolution is investigated, we should put $\eta_0 = \rho_0 e^{-i\Omega t}$ where $\rho_0$ and $\Omega$ are given, respectively, by Eq. (11) and (10). By keeping only linear terms in the evolution equations for plane waves, we obtain

$$\dot{\eta}_k = [1 - (1 + i\beta)\rho_0^2 - (1 + ie)k^2]\eta_k - 2(1 + i\beta)\rho_0^2 e^{-2i\Omega t}\eta^*_k. \quad (18)$$

Thus, so far as the linear stability analysis is performed, the problem is identical to an investigation of CGLE with external periodic forcing and corresponds to the case $n = 1$ of Refs. [27,28].

We see that, according to Eq. (18), pairs of modes with opposite wave vectors $k$ and $-k$ are coupled in the linear approximation and therefore all unstable modes should represent their symmetrical superpositions, i.e. standing waves. To investigate properties of such standing waves, it is more convenient to perform the linear stability analysis not for complex oscillation amplitudes but for variables $\rho(r, t)$ and $\phi(r, t)$. This immediately yields spatial distributions of the phase and the oscillation amplitude modulus in such a pattern (of course, the same results could be obtained after some algebraic transformations directly from equations for the amplitudes $\eta_k$).

When written for the variables $\rho(r, t)$ and $\phi(r, t)$, the model (9) gives a set of two equations:

$$\dot{\rho} = (1 - \rho^2)\rho + \nabla^2 \rho - \rho(\nabla \phi)^2 + \varepsilon \rho \nabla^2 \phi + 2\varepsilon \nabla \rho \nabla \phi + \mu R \cos(\phi - \Psi + \chi), \quad (19)$$
$$\dot{\phi} = \beta \rho^2 - \Omega + \frac{2}{\rho} \nabla \rho \nabla \phi + \nabla^2 \phi - \frac{\varepsilon}{\rho} \nabla^2 \rho + \varepsilon(\nabla \phi)^2 - \frac{\varepsilon}{\rho} \sin(\phi - \Psi + \chi). \quad (20)$$

Linearizing these equations near uniform oscillations with $\rho = R = \rho_0$ and $\phi = \Psi$, evolution equations for perturbations $\delta \rho(r, t)$ and $\delta \phi(r, t)$ are obtained:

$$\delta \dot{\rho} = -2\rho_0 \delta \rho + \nabla^2 \delta \rho + \varepsilon \rho_0 \nabla^2 \delta \phi - \mu \rho_0 \delta \phi \sin \chi, \quad (21)$$
$$\delta \dot{\phi} = 2\beta \rho_0 \delta \rho + \nabla^2 \delta \phi - \frac{\varepsilon}{\rho_0} \nabla^2 \delta \rho - \mu \delta \phi \sin \chi + \frac{\mu}{\rho_0} \delta \rho \cos \chi. \quad (22)$$

Solutions of these equations are linear superpositions of standing waves,

$$\delta \rho(x, t) = \delta \rho_k \exp(y_k t) \cos(kx), \quad (23)$$
$$\delta \phi(x, t) = \delta \phi_k \exp(y_k t) \cos(kx), \quad (24)$$

where the coordinate axis $x$ is chosen to be oriented along the vector $k$. The increments of growth $y_k$ satisfy the equation

$$(y_k^2 + 2 + 3\mu \cos \chi + k^2)(y_k^2 + \mu \cos \chi + k^2) + (\varepsilon k^2 + \mu \sin \chi)[2\beta(1 + \mu \cos \chi) + \varepsilon k^2 + \mu \sin \chi] = 0. \quad (25)$$

Depending on the parameters and the wave number $k$, this equation may have either two real or two complex conjugated roots $y_k$ (in the latter case the complex conjugate terms should be added to the right-hand sides of Eqs. (23) and (24) to obtain real quantities). Variations of the phase and the modulus in a growing standing wave mode are related as

$$\delta \phi_k = -\frac{y_k + 2\rho_0 + k^2}{\varepsilon \rho_0 k^2 + \mu \rho_0 \sin \chi} \delta \rho_k. \quad (26)$$
We first consider the situation where both roots $\gamma_k$ of the characteristic equation (37) are real. Then the instability threshold of uniform oscillations is reached when the largest of the roots for the modes with all possible wave vectors $k$ crosses zero and becomes positive. It means that at the threshold the conditions

$$
\gamma_k = 0, \quad \frac{\partial \gamma_k}{\partial k} = 0, \quad \frac{\partial^2 \gamma_k}{\partial k^2} < 0
$$

(27)

should be satisfied for a mode with a certain wave vector $k = k_0$.

Differentiating Eq. (25) in respect to variable $q = k^2$ and using conditions (27), the critical intensity of global coupling is determined as

$$
\mu_c = \frac{1 + \varepsilon \beta + (1 + \varepsilon^2)k_0^2}{(2 + \varepsilon \beta) \cos \chi + \varepsilon \sin \chi}.
$$

(28)

The wave vector $k_0$ of the first unstable mode is then found from Eq. (25), taking into account that $\gamma_k = 0$ at $k = k_0$ and that the intensity of global feedback at the threshold should be given by Eq. (28).

Note that the first unstable mode is in this case a standing wave. Indeed, it has the same frequency as that of the uniform oscillations (since $\text{Im} \gamma_k = 0$) and represents a growing periodic spatial variation (see (23) and (24)) of the phase and the amplitude modulus of such oscillations.

Although in general the solutions for $k_0$ and $\mu_c$ are complicated and can be only numerically analyzed, simple expressions for these properties are available near the BF boundary where the parameter $\nu = -1 - \varepsilon \beta$ is small. By keeping only the terms up to the second order in $\nu$, we find

$$
k_0 \approx \frac{\nu}{1 + \varepsilon^2} - \frac{\varepsilon v^2(\cos \chi + \varepsilon \sin \chi)}{2(1 + \varepsilon^2)^2(\varepsilon \cos \chi - \sin \chi)},
$$

(29)

$$
\mu_c \approx \frac{\varepsilon v^2}{2(1 + \varepsilon^2)(\varepsilon \cos \chi - \sin \chi)}.
$$

(30)

In the limit of small $\nu$'s variations of the phase and of the modulus in the first unstable standing wave mode are related as

$$
\delta \phi \approx -\frac{2(1 + \varepsilon^2)}{\varepsilon v} \delta \rho.
$$

(31)

It is interesting to compare the wavelength of the first unstable mode in the presence of global coupling with the characteristic size of the cells found in the same system in absence of global coupling, in the state of phase turbulence. Though the cells found in phase turbulence are not regular and their sizes vary with time and in space, there is a certain typical size of a cell determined by the wavelength of the most unstable perturbation mode of the uniform state. As follows from Eq. (25) at $\mu = 0$, the increment of growth of perturbations in the form of plane waves with a wave vector $k$ is given by

$$
\gamma_k = -1 - k^2 + \sqrt{1 - 2\varepsilon \beta k^2 - \varepsilon^2 k^4}.
$$

(32)

It reaches a maximum at the characteristic Kuramoto-Sivashinsky wave number $k_{KS}$ that satisfies the equation

$$
\varepsilon k_{KS}^2 = -\beta - \frac{1 + \beta^2}{1 + \varepsilon^2}.
$$

(33)

Near the BF boundary this yields

$$
k_{KS}^2 \approx \frac{\nu}{1 + \varepsilon^2} - \frac{\nu^2}{2(1 + \varepsilon^2)^2},
$$

(34)
where \( v = -1 - \varepsilon \beta \) is a small parameter \( (v \ll 1) \). The typical size of cells in phase turbulence is \( \lambda_{KS} = 2\pi/k_{KS} \); it goes to infinity when the BF boundary is approached.

We see that in the linear order in the parameter \( v \), specifying the distance from the BF boundary, the wavelength \( \lambda_0 = 2\pi/k_0 \) coincides with the characteristic size \( \lambda_{KS} \) of the cells in the phase turbulence regime of CGLE (cf. Eqs. (34) and (29)) and does not depend on the phase shift parameter \( \chi \). The difference between the two characteristic lengths is revealed in the second-order corrections, already sensitive to the phase shift. The critical intensity \( \mu_c \) of global coupling is proportional in this limit to \( v^2 \). According to Eq. (31), variations of \( \rho \) and \( \phi \) in the standing wave are shifted by half of the spatial period, i.e. the phase \( \phi \) has its maxima where the modulus \( \rho \) is minimal.

Fig. 1 shows the dependence of \( k_0^2 \) as function of \( v \), obtained by numerical solution of Eqs. (25) and (27), together with its quadratic approximation (29). We see that though this approximation has been derived assuming that \( v \ll 1 \), it remains satisfactory even at larger values of this parameter.

The instability threshold \( \mu_c \) rapidly increases as the phase shift \( \chi = \arctan \varepsilon \) is approached.\(^2\) This is accompanied by a decrease of the wave number \( k_0 \) of the first unstable standing wave. As seen from (29), this wave number reaches zero at certain point \( \chi^* \) shortly before \( \mu_c \) diverges. Above this point, Eqs. (25) and (27), we find that for \( \chi > \chi^* \) they have a solution where the maximal increment of growth is achieved at \( k = 0 \) and therefore the first unstable mode has \( k_0 = 0 \). In this region the instability threshold is

\[
\mu_c = -\frac{2(\beta \sin \chi + \cos \chi)}{1 + 2 \cos \chi (\beta \sin \chi + \cos \chi)}.
\]

This branch meets at \( \chi = \chi^* \) the branch with \( k_0 \neq 0 \), described by Eqs. (29) and (30). When \( \chi > \chi^* \), the increment of growth \( \gamma_k \) is minimal at \( k = 0 \) for \( \mu \) given by Eq. (35).

\(^2\) Here and below, unless specially noted, we consider the phase shifts inside the interval \( -\pi < \chi < +\pi \). We restrict our analysis to the case of positive values of the parameter \( \varepsilon \).
A note of caution should now be made. As follows from Eq. (15), the uniform mode with \( k = 0 \) is always stable in the linear approximation. The characteristic equation (25) is derived by considering small nonuniform perturbations of the uniform oscillations. Therefore, any unstable mode determined by this equation must have a nonvanishing wave number. If periodic boundary conditions are used, the minimal wave number is \( k_{\text{min}} = \frac{2\pi}{L} \) where \( L \) is the spatial size of the medium.

For large systems \( L \to \infty \) and therefore the limit \( k \to 0 \) can be considered. Thus, when modes with \( k = 0 \) are considered, we actually mean the modes with the minimal possible wave number, specified by the size of the medium and the boundary conditions. In our numerical simulations, described below, we see that, when such an instability occurs, the medium breaks into large domains whose sizes are about the dimension of the medium.

Finally, unstable modes with \( \text{Im} \gamma_k \neq 0 \) are also possible. As follows from Eq. (25), for such modes we have

\[
\text{Re} \gamma_k = -1 - k^2 - 2\mu \cos \chi.
\] \tag{36}

Therefore, the least stable mode is always the mode with \( k = 0 \) (i.e. it has the maximal possible wavelength for a given medium). The instability occurs only if the \( \cos \chi < 0 \). In contrast to the instabilities related to growth of standing waves, this instability develops under an increase of the feedback intensity \( \mu \), i.e. when \( \mu > \mu_0 \) where

\[
\mu_0 = \frac{1}{\cos \chi}.
\] \tag{37}

Therefore, for some phase shifts \( \chi \) uniform oscillations may be stable inside an interval of the global feedback intensity \( \mu \). They become stabilized as a certain low boundary \( \mu_c^{(1)} \), given by Eqs. (30) or (35), is crossed but loss again their stability a higher value \( \mu_c^{(2)} \), given by Eq. (37), and are then replaced by turbulence.

Fig. 2 shows the stability diagram of uniform oscillations in the plane \((\chi, \mu)\) which has been obtained by direct numerical solution of the characteristic equation (25). In this diagram the parameters \( \varepsilon = 2 \) and \( \beta = -1.4 \) has been chosen, so that the parameter \( v = -1 - \varepsilon \beta \) is no longer small \((v = 1.8)\). Uniform oscillations are stable above the boundary consisting of curves AB, BC, CE and AD.

When the curve AB is crossed, standing waves with \( k_0 \neq 0 \) appear. The dependence of their wavelength \( \lambda_0 = \frac{2\pi}{k_0} \) on the phase shift \( \chi \) is shown in Fig. 3 (the horizontal line in this figure shows the respective typical size \( \lambda_{KS} \) of the cells in the phase turbulence regime in absence of the global feedback). This wavelength diverges as the point B, corresponding to \( \chi^* \approx 0.275\pi \), is approached. The wavelength monotonously decreases when the phase shift is decreased.

When the curve BC, determined by Eq. (35), is crossed, a large-scale domain structure develops in the medium, with the characteristic size of domains about the size of the entire medium. This domain structure has at the threshold the same frequency as that of the uniform oscillations and is therefore ‘standing’ or static, in the coordinate frame rotating with the frequency of the uniform oscillations.

In contrast to this, if curves DA or CE are crossed, long-wavelength oscillatory perturbations start to grow. These two curves asymptotically approach \( \mu = \infty \) at \( \chi = \pm \pi/2 \). For \( \chi_- < \chi < -\pi/2 \) and for \( \pi/2 < \chi < \chi_+ \), where \( \chi_- \approx -0.6235\pi \) and \( \chi_+ \approx 0.599\pi \), uniform oscillations are stable only inside certain intervals of the feedback intensity \( \mu \).

Of course, the linear stability analysis does not yet allow to determine, whether a bifurcation is supercritical or subcritical. This can only be found by investigating the role of nonlinear terms, giving rise to interactions between the modes. When an instability of uniform oscillations leads to growth of a mode with \( k_0 \neq 0 \), the characteristic equation (25) fixes only the magnitude \(|k| = k_0 \) of the wave vector, but leaves its direction arbitrary. Therefore, all
such modes grow in the linear approximation. Nonlinear interactions may select a group of modes that eventually dominates over the medium. Due to mutual enhancement, the modes in this group can grow even when single modes already die out. This changes the nature of the bifurcation, making it subcritical and characterized by a hysteresis.
4. Destruction of phase flips

We consider phase flips that appear in CGLE under negative global feedback. Inside such a flip the oscillation phase makes full $2\pi$ rotation. These objects travel at a constant velocity preserving their spatial profile. Phase-flip solutions are constructed below using a reduced phase description for the system with global feedback. The BF-stable case where the condition $1 + \varepsilon\beta > 0$ holds and local diffusional coupling tends to stabilize uniform oscillations is here considered.

Suppose that the local oscillation phase $\phi(r, t)$ slowly varies in space and in time and that the modulus $\rho(r, t)$ of the oscillation amplitude only slightly deviates from its constant value $\rho_0$ characteristic for uniform oscillations. Then the modulus adiabatically adjusts to the local spatial derivatives of the phase, i.e.

$$\rho = \rho_0 + \frac{\varepsilon}{2} \nabla^2 \phi - \frac{1}{\rho_0} (\nabla \phi)^2,$$

and the phase description is applicable [3,29]. In this approximation evolution of the phase variable obeys the equation [22]:

$$\dot{\phi} = g(\phi - \Psi) + (\varepsilon - \beta)(\nabla \phi)^2 + (1 + \varepsilon\beta)\nabla^2 \phi$$

(39)

that differs from the usual phase dynamics equation for CGLE (see [17]) by the presence of an additional term,

$$g(\phi - \Psi) = \mu \sin \chi_1 - \frac{R}{\rho_0} \sin(\phi - \Psi + \chi_1),$$

(40)

taking into account action of global feedback and depending on the difference between local oscillation phase $\phi$ and global phase $\Psi$. Assuming that $\mu \ll 1$ we keep only the terms up to the first order in $\mu$; the notation $\chi_1 = \chi - \arctan \beta$ is employed.

Depending on the phase shift $\chi$, the feedback can be either positive or negative, i.e. it can destroy uniform oscillations or increase their stability. To find the conditions for a negative global feedback, behavior of small local perturbations of the uniform state should be considered.

Suppose that the local oscillation phase is slightly shifted ($\phi = \Psi + \delta\phi$) in some region in respect to the global oscillation phase $\Psi$, fixed by uniform oscillations in the rest of the medium. This region is small in comparison to the total size of the medium but larger than the diffusion length, so that the terms with spatial derivatives in (39) can be neglected. Evolution of such perturbations is described by the linearized equation

$$\frac{d\delta\phi}{dt} = g'(0)\delta\phi,$$

(41)

where

$$g'(0) = \left. \frac{dg}{d\phi} \right|_{\phi=\Psi} = -\mu \sqrt{1 + \beta^2 \cos \chi_1}.$$

(42)

We see that the sign of the derivative $g'(0)$ determines whether a perturbations would grow or fade.

If $\cos \chi_1 < 0$ (or, explicitly, $\cos \chi + \beta \sin \chi < 0$), global feedback is positive and the local phase tends to deviate from the global phase. Numerical simulations show that in this case the medium breaks into a few large domains with different oscillation phases [22]. This situation is reminiscent of spontaneous breaking of ferromagnets into individual magnetic domains, leading to disappearance of the total magnetization. We find below that similar behavior is also observed in the BF-unstable region.

When negative global feedback is realized (i.e. $\cos \chi_1 > 0$), small local phase variations are damped. In this case, however, travelling phase flips are possible [21,22]. If a large medium contains a single phase flip, its presence
does not significantly influence the global oscillation phase $\psi$ determined by uniform oscillations in the rest of the medium and, moreover, $R \approx \rho_0$. It can be said that the rest of the medium generates in this case a periodic driving force. Hence, the situation is similar to the problem with external forcing where phase flips have earlier been considered [27,28]. An important difference is, however, that the driving force produced by a global feedback is always resonant.

The phase dynamics equation (40) has an infinite number of phase-locked states with phases $\phi_m = \psi + 2\pi m$, where $m = 1, 2, 3, \ldots$. Though such states are physically indistinguishable, travelling waves of transitions between them are possible. They represent phase flips

$$\phi(x,t) = \Phi(\xi), \quad \xi = x - Vt,$$

satisfying the conditions $\phi \to \psi$ for $\xi \to \infty$ and $\Phi \to \psi + 2\pi$ for $\xi \to -\infty$. Inside a flip the phase undergoes a full $2\pi$ rotation; $V$ is the propagation velocity of the flip.

To construct these solutions and analyze their properties, we perform in Eq. (40) a nonlinear transformation to the new variable $u$, defined by

$$\phi - \psi = \frac{1 + \epsilon \beta}{\epsilon - \beta} \ln u,$$

which yields for the variable $u$ the evolution equation

$$\dot{u} = Q(u) + (1 + \epsilon \beta) \nabla^2 u$$

with the nonlinear source function given by

$$Q(u) = \frac{\mu u}{1 + \beta^2} \left[ \frac{\epsilon - \beta}{1 + \epsilon \beta} \sin \left( \chi_1 + \frac{1 + \epsilon \beta}{\epsilon - \beta} \ln u \right) - \sin \chi_1 \right].$$

An equation of the same form describes propagation of trigger waves in bistable one-component reaction–diffusion systems [17] and results of the respective analysis are directly applicable here.

Eq. (45) has traveling wave solutions $u = U(\xi)$ with $\xi = x - Vt$ that satisfy asymptotic conditions

$$U(\xi) \to u_1 \text{ for } \xi \to -\infty \quad \text{and} \quad U(\xi) \to 1 \text{ for } \xi \to \infty,$$

where

$$u_1 = \exp \left[ \frac{2\pi (\epsilon - \beta)}{(1 + \epsilon \beta)} \right].$$

This solution, including its propagation velocity $V$, is uniquely determined by conditions (47). Particularly, the velocity sign is controlled by the integral (cf. [17])

$$J = \int_{1}^{u_1} Q(u) \, du.$$

Calculating this integral, we find after some transformations that an important role is played by the parameter combination

$$A = \sin \chi_1 + \frac{2(\epsilon - \beta)}{(1 + \epsilon \beta)} \cos \chi_1.$$

The velocity $V$ is positive when $A < 0$ and negative if $A > 0$. Note that for the solution (47) the positive propagation velocity means that phase $\phi$ is increased by $2\pi$ after a phase flip has passed through a given point of the medium; it
is decreased by $2\pi$ if $V$ is negative. In other words, the phase inside a phase flip may rotate either in the clockwise or the counterclockwise direction, depending on the sign of $A$. The phase flips are standing if $A = 0$.

Though the variable $u$ changes from 1 to $u_1$ inside a propagating phase flip, these values correspond to a phase change of $2\pi$ and therefore the states of the medium before and after the passage of the flip are physically identical. The pattern is therefore visible only in a narrow region where the phase varies.

The width of a phase flip is estimated from Eq. (46) as

$$
\delta x \approx \left[ \frac{1 + \varepsilon \beta}{\mu (\varepsilon - \beta)(1 + \beta^2)^{1/2}} \right]^{1/2}.
$$

(51)

It is inversely proportional to the square root of the feedback intensity $\mu$. The phase gradient inside a phase flip can be approximately estimated as $\nabla \phi = 2\pi/\delta x$. Hence, the phase flips which we consider in this section are essentially related to the presence of a global feedback in the system. They disappear when the intensity $\mu$ of this feedback goes to zero.³

According to Eq. (38), variation of the phase leads to variation of the modulus $\rho$ inside a phase flip. Taking as the center of the flip the point $x$ where $\partial^2 \phi / \partial x^2 = 0$, we find that the amplitude modulus is decreased there by

$$
\delta \rho = -\frac{2\pi^2 \mu (\varepsilon - \beta)(1 + \beta^2)^{1/2}}{1 + \varepsilon \beta}.
$$

(52)

This can be taken as the characteristic magnitude of the modulus variation in a phase flip. The amplitude depression in a flip becomes stronger when the BF boundary $1 + \varepsilon \beta = 0$ is approached. It also increases for higher feedback intensities $\mu$.

Note that the reduced phase description, i.e. Eq. (38), is applicable while $\delta \rho \ll 1$. To investigate the behavior of phase flips at higher feedback intensities, numerical simulations based on Eq. (9) have been performed. These simulations have been carried out for the one-dimensional medium of length $L = 102.4$ with periodic boundary conditions, using a small coordinate step $dx = 0.1$ and a short time step $dt = 10^{-4}$ (such small step is necessary because $\varepsilon = 5.0$ in this simulation). To create a phase flip, we used periodic boundary conditions and started with the initial condition where the modulus of the oscillation amplitude was constant over the medium but the phase gradually decreased by $2\pi$ from the left to the right-hand sides of the interval (i.e. the winding number was equal to 1).

Fig. 4 displays phase and modulus profiles in a traveling phase flip at a moderate intensity of global feedback. This object is stable and travels at a constant velocity, though the modulus depression in it is already considerable. Note also that the depression of the amplitude modulus is actually preceded by its small increase.

Numerical simulations have shown that further increase of the feedback intensity leads to destruction of phase flips which disappear transforming to uniform oscillations. When the simulation is started from the described above initial condition, the profile similar to that of a steadily traveling phase flip (Fig. 4) first appears. Then the flip slightly accelerates and the amplitude depression in its center increases ($T = 112.5$ in Fig. 5(a)). At the same time, the phase profile gets steeper ($T = 112.5$ in Fig. 5(b)). Within a very short time, the amplitude in one point inside the flip region sharply falls down and a jump develops in the phase profile. In the frame $T = 113.07$ of Fig. 5(a) the minimal value of the amplitude modulus is $\rho = 0.0012$. The respective phase distribution, shown at $T = 113.07$ in Fig. 5(b), displays a large phase jump. Since integration is performed with a small, but still finite time step, we cannot see the exact moment when the amplitude vanishes (i.e. $\rho = 0$). However, it seems that such a moment exists and lies very close to the time $T = 113.07$.

³ Hence, these phase flips are different from the shock-hole-shock structures in CGLE without global feedback [34,35].
If the oscillation amplitude vanishes in a certain point, the oscillation phase is not defined there. The phase distributions in the regions lying on the left and right-hand sides of this point become then disconnected and the phase can be changed by $2\pi$ while passing this point. Thus, the part of the phase distribution on the left-hand side of the singularity point may be shifted down by $2\pi$, so that the phase levels at both ends $x = 0$ and $x = L$ of the medium become equal.

At a later moment, the amplitude modulus is again everywhere positive, though it has a strong depression near the point where it has reached zero (frame $T = 113.22$ in Fig. 5(a)). The phase slowly decreases as this point is approached and then sharply grows, forming a local maximum. Farther away from the former singularity point, the phase slowly returns to its initial value (frame $T = 113.22$ in Fig. 5(b)). Subsequently, this local pattern smears out (frame $T = 114.5$ in Fig. 5) and uniform oscillations are established.

Thus, the winding number is not conserved in this one-dimensional system in the presence of strong global feedback. Destruction of phase flips proceeds through the formation of a phase singularity. Fig. 6 displays a space–time diagram for the above described process of disappearance of the phase flip. Here the dash lines represent the trajectories of the points where $\text{Im} \gamma(x, t) = 0$, whereas the bold lines show the trajectories intersect. The intersection point, where $\rho = |\eta| = 0$, is the point of this phase singularity.

Though the profile of the modulus distribution in the phase flip shortly before its disappearance is qualitatively similar to that of the slowly moving Bekki–Nozaki (BN) hole [7,14,30,31], the phase distribution is different. The phase shift in a BN hole is close to $\pi$, whereas the phase shift in a phase flip is $2\pi$. Furthermore, a BN hole is a source of plane waves, whereas behind and after of a phase flip uniform oscillations are found. It means that destruction of a phase flip does not proceed through the formation of a BN hole and represents a different dynamical process.

In two-dimensional media, curved traveling waves of phase flips are possible. As intensity of global feedback is increased, destruction of these waves, similar to the described above process in a one-dimensional system, occurs (Fig. 7). Modulation of this wave first develops, so that in some part of the wave the phase variation becomes condensed inside a very narrow stripe which we call a string. Since the oscillation amplitude is greatly decreased inside the string, it can be classified as an extended amplitude defect. The thickness of the string varies with time. At one of its points, the string becomes infinitely thin, the oscillation amplitude vanishes at this point and the string breaks up, so that two open ends are formed. The remaining parts of the string then shrink and the rupture grows, so that the open ends move further away one from another.
Fig. 5. Destruction of a phase flip by strong global feedback ($\varepsilon = 5.0, \beta = 0.5, \chi = 0, \mu = 0.175$). Spatial profiles (a) of the amplitude modulus and (b) of the phase $\phi/\pi$ at subsequent time moments $T$ are shown.
Fig. 6. Space–time diagram showing destruction of a phase flip. The solid curve is the trajectory of the point with \( \text{Re} \, \eta(x, t) = 0 \), the dash curve is the respective trajectory for \( \text{Im} \, \eta(x, t) = 0 \). A phase singularity appears at the moment when the two curves intersect; \( \rho = 0 \) in the intersection point. The same parameters as in Fig. 5.

Fig. 7. Destruction of a curved two-dimensional phase-flip wave by strong global feedback. The upper row shows spatial distribution of the modulus \( \rho \) of the complex oscillation amplitude in the medium of size 100 \( \times \) 100 at three subsequent time moments \( T \) for \( \varepsilon = 5.0 \), \( \beta = 0.5 \), \( \chi = 0 \), and \( \mu = 0.18 \). The bottom row shows the contours \( \text{Re} \, \eta(r, t) = 0 \) (solid curves) and \( \text{Im} \, \eta(r, t) = 0 \) (dash curves) in the medium at the same time moments. Topological defects lie in the points of intersection of these contours.
At an open end of the broken string, a phase singularity is located and the oscillation amplitude vanishes. The phase changes by $2\pi$ after going around any contour, surrounding the open end, and therefore a topological defect should be located in this region. However, its properties are significantly different from that of a topological defect lying in the center of a rotating spiral wave. Now the topological defect is attached to an end of an extended amplitude defect, i.e. to the string. Hence, destruction of a phase-flip wave in the two-dimensional oscillatory medium is similar to the break-up of waves in excitable systems [14]. The tip of the excitation wave corresponds now to the end-point singularity of a broken phase-flip wave.

The above analysis of phase flips has been performed in the BF-stable region, i.e. where the condition $1 + \varepsilon \beta > 0$ holds. However, similar objects are found below in numerical simulations in the parameter region where the opposite condition is satisfied and uniform oscillations are unstable in absence of global feedback.

5. Destruction of spiral waves

On the right-hand side of the line $T$ in the phase diagram (Fig. 8) frozen states are observed in the system without global feedback. In such states the medium is filled by a number of steadily rotating spiral waves. These spiral waves develop after a transient from the chaotic state, which first appears when the initial condition $\eta = 0$ with small random perturbation is used. In our simulations we have investigated evolution of such spiral waves as global feedback described by Eq. (9) is introduced and its intensity $\mu$ is gradually increased. Fig. 9 shows destruction of spiral waves by increasing global feedback. The solid curve in the upper part of this figure displays the time evolution of the synchronization parameter $R$ defined by Eq. (14); the dash line indicates the respective (linear)
increase of the feedback intensity $\mu$ with time. In the lower part of the figure, characteristic spatial distributions of the variable $\cos \phi$ at six subsequent moments are shown. Parameters $\varepsilon$ and $\beta$ in this simulation correspond to the point 1 in the phase diagram of Fig. 8.

While $\mu$ is small, several spiral waves are present. Then one of these waves begins to dominate and spreads its activity over the entire medium. When the spiral wave covers the whole medium, the synchronization parameter $R$ is close to zero because different parts of this pattern oscillate with different phases. Inside the synchronization window, the growing global feedback tends to establish synchronous oscillations and thus suppress the spiral wave. This occurs in the following way: The waves, traveling from the core of the spiral wave, become destroyed at a certain distance from its center and are replaced there by relatively uniform oscillations. As a result, the rotating spiral is confined inside a roughly circular spatial region surrounded by the area with synchronous oscillations. This region slowly shrinks and the process is accompanied by growth of the parameter $R$, showing a higher degree of synchronization in the medium.
We sometimes observed that, depending on the choice of the system’s parameters, spiral waves spread again its influence over almost the entire medium and thus destroyed the synchronization. Then the synchronization parameter $R$ was decreased. This resembled the intermittency found in numerical simulations [32] of CGLE without global feedback – the difference was, however, that the area outside of the spiral wave was occupied not by the defect turbulence, but by almost uniform oscillations.

As the intensity of global feedback is further increased, the spiral wave becomes localized inside a small circular region which quickly shrinks and disappears. This stage is characterized by monotonous growth of the synchronization parameter $R$ until uniform oscillations are established in the entire medium. If the global feedback intensity $\mu$ is increased at a high rate, the intermittent stage is not found and the spiral wave dies immediately out.

Similar evolution under increasing of global feedback was found in the numerical simulations for a different choice of the parameters (point 2 in Fig. 8) where, in absence of global feedback, a large number of rotating spiral waves was observed. As displayed in Fig. 10, a population of spiral waves responds to an increase of global feedback by confining the activity of individual spiral waves inside islands bounded by certain ‘membranes’. Outside of these islands, the oscillations are approximately uniform. The islands populated by wave fragments are not stationary – they change their shapes, move through the medium and can merge. As the intensity of global coupling grows, the islands get smaller and acquire more regular circular shapes. Spiral fragments disappear from the interior of the islands and they collapse, giving way to synchronous oscillations.

It should be noted that, though this phenomenon has not been specifically investigated, spiral waves rotating inside islands surrounded by uniform oscillations have earlier been described in an experiment [33] with catalytic surface chemical reactions, where the global feedback was intrinsically present because of the global coupling via the gas phase.

To analyze in more detail the processes that accompany destruction of a spiral wave by increasing global feedback, we show in Fig. 11 the spatial distributions of (a) the oscillation amplitude modulus $\rho$ and (b) the phase $\phi$ at subsequent time moments during the process. No-flux boundary conditions have been used in this simulation. The speed of increase of the feedback intensity is now higher than in Fig. 9 and the intermittency does not develop. The
Fig. 11. A detailed view of the process leading to the destruction of a spiral wave by increasing global feedback ($\varepsilon = 2.0$, $\beta = -0.6$, $\chi = 0$, $\mu$ increases at a constant speed from 0.045 to 0.05). Subsequent frames, corresponding to different moments $T$, show in gray scale the spatial distributions of (a) the oscillation amplitude modulus and (b) the oscillation phase.

contrast in the frames of Figs. 11(a) and (b) is increased, so that variations of the modulus and the phase are better discernible for small values of these variables.

The frame $T = 0$ in Fig. 11 displays a spiral wave in absence of global feedback ($\mu = 0$). The modulus $\rho$ is significantly decreased only inside a small core region of the rotating wave and reaches zero in its center. Outside of this region the variable $\rho$ is practically constant.

The black–white spiral interface in the spatial distribution of the oscillation phase in the frame $T = 0$ indicates the line, along which the phase reaches the value $2\pi$ (white area) and switches to the value 0 (black area). This sharp interface is a consequence of the definition used for plotting of the phase variable. It does not correspond itself to any physical singularity. This form of visualization is, however, convenient: if a closed contour crosses such an interface, the phase increment along this contour is equal to $2\pi$. The interface ends in the central point where the phase singularity is located.

When the global feedback is switched on, the properties of a spiral wave are changed. As seen in the frame $T = 73$, the spiral is visible now not only in the phase distribution, but also in the spatial distribution of the amplitude modulus. Inside a narrow curled dark stripe, repeating the shape of the phase front, the modulus $\rho$ is somewhat decreased. Examination of the figure also reveals that the dark front is preceded by a slightly brighter area with an increased oscillation amplitude. This is similar to the profile of $\rho$ in the spatial cross-section of a traveling phase flip (Fig. 4). Therefore, we approximately describe this object as a spiral wave formed by a phase-flip wave. Though the amplitude modulus is diminished inside the phase flip, forming the arm of a spiral wave, it vanishes only in one point in the center of the spiral wave, where the phase singularity is located. The other end of the phase flip reaches the boundary of the medium.

At a later moment, the outer part of the phase flip, rotating along the system boundary, becomes destroyed. This is preceded by the development of a front instability, already visible at $T = 73$. We see in this frame that the
amplitude modulus $\rho$ gets smaller in the front close to the boundary. Moreover, its distribution along the front is here nonuniform, i.e. at some points on the front the values of $\rho$ are lower. Later, the amplitude vanishes in these points, the phase singularity develops and the phase flip breaks, similar to what has been described in Section 4.

At the moment $T = 80$, destruction of the phase front in the periphery region is already visible. In the frame, showing the spatial phase distribution at this moment, the sharp interface is absent in the area adjacent to the right boundary of the medium. In this entire area the phase does not reach the value $2\pi$.

At $T = 86$ we see a long black curled string that begins approximately at the end of the sharp phase boundary and extends, along the phase flip front, towards the center of the rotating spiral. Inside this string, the local level of the oscillation amplitude modulus is strongly reduced, though it does not yet reach zero. The point where the sharp black–white boundary for the phase distribution ends should correspond to a phase singularity and the oscillation amplitude must vanish there. By comparing spatial distributions of the phase and the modulus in this region, we notice that the modulus $\rho$ indeed becomes very small here, i.e. the local black level is the same as that characteristic for the core of the spiral wave. However, in contrast to the core, a strong depression of the amplitude modulus is not localized near a certain point. It represents an extended amplitude defect which contains a topological defect, i.e. the phase singularity point, near its end.

When such a string has just appeared, it is relatively short and does not come close to the center of the spiral, where the profile characteristic for a phase flip with a smaller magnitude of the amplitude reduction persists. However, the string moves closer to the core of the spiral ($T = 88$) and, at the same time, its back retrieves, leaving after it the region with approximately uniform oscillations (note that the string is followed by a long curled bright tail with the increased oscillation amplitude).

At a certain moment ($T = 90$) the string enters the core of the spiral wave. The phase distribution at this time moment shows that the pattern characteristic for a rotating spiral wave is visible only near the center of the medium, and it is surrounded by the area with the approximately uniform oscillation phase. Later the string forms a shortening black tail ($T = 92$).

At $T = 94$ a small elongated and slightly curved black drop is seen in the spatial distribution of the amplitude modulus in the center of the medium. The respective phase distribution shows a small dark area, separated by a sharp boundary from the area with uniform oscillations. This phase domain grows and the modulus of the oscillation amplitude inside it does not reach zero now ($T = 97$).

At the last stage, not shown in Fig. 11, a ring-shaped amplitude domain develops in the center of the medium. It is surrounded by a bright membrane, where the oscillation amplitude modulus is increased. The ring breaks into parts contained inside the membrane. These parts shrink and disappear, so that uniform oscillations are established in the entire medium.

Though the process, illustrated by Fig. 11, is recorded under a gradual increase of the global feedback intensity, this intensity $\mu$ varies only between 0.045 and 0.05 within the entire time interval of the process. A similar behavior leading to the destruction of a spiral wave has actually been found by us also in the simulations when this intensity has been kept at a fixed level. These simulations at a fixed feedback intensity have shown, moreover, that merging of the extended amplitude defect (i.e. the string) and the core of the spiral wave is a very important event. If this has not occurred, the spiral wave can spread again its activity over the medium, as found under the intermittent conditions.

6. Suppression of developed defect turbulence

Farther away from the BF boundary in the unstable region, the state of developed defect turbulence is reached. Our numerical simulations indicate that suppression of such defect turbulence by increasing global feedback is
Fig. 12. Suppression of defect turbulence via formation of a cellular structure ($\varepsilon = 2, \beta = -1.4, \chi = -0.2\pi$). The upper part shows the growth of the global feedback intensity $\mu$ (dash line) and the dependence of the synchronization parameter $R$ on time. The middle part displays the respective evolution of the spatial distribution of the amplitude modulus $\rho$ in the central vertical cross-section of the medium, dark areas correspond to smaller values of $\rho$. Three frames in the bottom row show spatial distributions of the variable $\rho$ in the entire medium at selected time moments $T = 200, 600, and 700$.

achieved by different scenarios, depending on the phase shift $\chi$ of the global feedback. Synchronous oscillations emerge through appearance of a spatial cellular structure, through the regime of localized turbulence, or through formation of large phase domains. In the simulations described below in this section we always start from the same initial state of defect turbulence (point 5 in Fig. 8) but choose different values of the phase shift for the ramped global feedback.

An example of evolution following the first scenario is given in Fig. 12. The dash line in the top part of this figure shows growth of the feedback intensity $\mu$, the solid curve presents the respective evolution of the synchronization parameter $R$. The middle part shows time development for the spatial distribution of the modulus $\rho$ of the local oscillation amplitude along the vertical spatial cross-section drawn through the center of the medium. At the bottom,
characteristic spatial distributions of the modulus in the entire medium at three different subsequent time moments are shown. The gray-scale coding is chosen, so that darker regions correspond to smaller values of $\rho$. Almost black small regions indicate amplitude defects where $\rho$ is close to zero. Boundaries of the cells are formed by shocks with an increased modulus of the oscillation amplitude, which look therefore like thin light membranes.

As the intensity of global feedback increases, shocks surrounding amplitude defects become less mobile and arrange into more regular cellular structures. Inside some cells, amplitude defects disappear and thus empty cells are produced. The cells populated by defects have larger sizes. Occasionally, the defects destroy a cell boundary formed by a shock and penetrate into neighboring cells. The competing influences of the global feedback, tending to impose synchronization, and of the defects, tending to spread turbulence, balance for some time each another, despite a steady increase of the feedback intensity. Inside this time interval, as seen from Fig. 12, the synchronization parameter $R$ remains near to 0.3.

Under an increase of $\mu$, the number of defects gets smaller and they gradually lose their ability to destroy the cell boundaries. Finally, the growing global feedback completely eliminates the defects and the medium becomes filled only with the hexagonal cells seen in the second frame in the bottom of Fig. 12. In the spatial cross-section, such cellular structures correspond to standing waves in the one-dimensional system.

As $\mu$ is further increased, the cell boundaries are gradually washed out and the amplitude variation between the center and the boundaries of a cell gets less pronounced. This process leads to an almost monotonous increase of the synchronization parameter $R$, until uniform oscillations are established and a constant value of $R$ is reached.

The second scenario of the synchronization onset is illustrated by Fig. 13 where a different value of the phase shift is taken. The growth of the synchronization parameter $R$ is accompanied now by large fluctuations. They are related to bursts in the number of defects, as seen in the time evolution in the central cross-section.

In comparison to the previously described case, a higher intensity of global feedback is needed here to establish uniform oscillations. Empty cellular structures, forming the background, dissolve now before the disappearance of the cells populated by defects has taken place. As a result, cells containing defects form clusters, or islands, on the background of uniform oscillations. Sizes and shapes of such turbulent islands vary in time, and they can move through the medium.

When the intensity of global coupling is further increased, these turbulent islands get smaller and the fraction of the medium covered by them decreases. Finally, all turbulent islands die out and uniform oscillations are established in the medium.

The third scenario is observed when we take the phase shift $\chi$ inside the interval between $\chi^*$ and $\chi_+$ in Fig. 2. In this case, the linear stability analysis predicts that, when the global feedback intensity is decreased, the uniform oscillations should first give rise to large phase domains. Note that this occurs at much large values of the feedback intensity $\mu$.

In the simulation shown in Fig. 14 we slowly increase the feedback intensity. The initial stage of the pattern evolution is similar to that found in the second scenario. A cellular structure first appears that later transforms into clusters of cells surrounded by the area with uniform oscillations. Later, however, the shock boundaries between the cells in a cluster dissolve and a large single domain is formed. Initially, this domain has complex shape and nonuniform amplitude activity is observed inside it. As the intensity of global feedback is further increased, the domain acquires an almost elliptical shape and its interior becomes approximately uniform. At still higher intensities, the domain shrinks and finally disappears.

Note that if we stop increasing the global feedback and fix its intensity, the phase domains are stable and this pattern is persistent. The modulus is slightly decreased inside such a steady domain. The phase slightly decreases from the center towards the boundary and then abruptly drops on the boundary reaching its level for uniform oscillations in the outside region.
Numerical simulations performed under gradual increase of the global feedback intensity, which have been mainly described above, yield a general view of the processes leading to the synchronization onset in the medium. Below we investigate in more detail the typical properties of appearing patterns under the conditions when the global feedback intensity and other parameters of the system are kept constant.

7. Cellular structures

Cellular structures are formed by triplets of standing wave modes, whose wave vectors satisfy the condition $k_1 + k_2 + k_3 = 0$ and $|k_1| = |k_2| = |k_3|$. Nonlinear effects saturate growth of these modes and thus a steady hexagonal cellular structure becomes formed. Note that the pattern, that develops from random initial conditions or
when going from the turbulent state in a ramping experiment, is very irregular. Long time should pass until a slow relaxation process leading to ordering of the cellular pattern is finished.

In contrast to the Benard or Turning instabilities, cellular structures in the considered oscillatory system are not stationary. Fig. 15 shows the pattern evolution during a single oscillation period at a fixed intensity of the global feedback. Here the spatial distribution of the local oscillation phase $\phi$ at subsequent time moments displayed. A cycle begins with appearance of a relatively regular array of islands. They slowly grow and, shortly before they merge, a hexagonal network is clearly seen. This network is further discernible on the dark background until it fades out and the oscillation cycle is repeated. Cellular structures with such properties have recently been experimentally observed in oscillatory catalytic surface reactions [25].

Note that the dynamical pattern formed by a cellular structure is therefore similar to that of the phase turbulence. The differences are that the cells forming this structure have equal sizes and are ordered into a hexagonal array.
Fig. 15. A cellular structure ($\epsilon = 2, \beta = -1.4, \chi = -0.2\pi$, and $\mu = 0.245$). Spatial distributions of the oscillation phase $\phi$ are shown at six subsequent time moments within a single oscillation cycle. The phase increases from 0 to $2\pi$ and then jumps back to 0, the black fronts display the lines of the constant phase $\phi = 2\pi$.

Though the cells result from nonlinear interactions, insight into their properties (later confirmed by numerical simulations) can be gained from the linear theory. As follows from Eq. (31), in a single standing wave mode the oscillation phase $\phi$ reaches its maximum at a point where the amplitude modulus $\rho$ is minimal. When a superposition of three such modes, representing a cellular structure, is constructed, this point would correspond to the center of a cell. Since the phase is maximal there, oscillations originate in the center and spread towards the cell’s boundary. On the boundary, oscillation fronts coming from two neighboring cells meet and annihilate. Thus, the cells behave as if there were a pacemaker sitting in each of their centers. Note, however, that the phase gradients inside a cell are small, i.e. the cell’s size is shorter than the wavelength of produced waves. As a result, only single fronts spreading from the center and annihilating at the boundary are seen. Since waves are colliding at the boundary, a shock-like object can be expected there. Indeed, according to Eq. (31), the modulus is increased where the phase has its minimum, i.e. on the boundary of a cell.

We performed systematic numerical studies of cellular structures. Because the system may possess several global attractors in the considered parameter region, only one of which corresponds to a cellular structure, the following simulation procedure was adopted: By going from the region of developed defect turbulence, the feedback intensity $\mu$ was gradually increased until a cellular structure first appeared. Then we fixed the respective value of $\mu$ and continued integration to check whether the observed structure was persistent. If a stable structure was found, it was used as the initial condition for the next simulation where the parameters $\mu$ and $\chi$ were slightly changed. If this again yielded a stable structure, it was used as the initial condition for the next step, etc. In this manner, boundaries for the existence of stable cellular patterns and other steady spatiotemporal regimes have been determined.

Results of a large series of numerical simulations are summarized in Fig. 16. The solid curve in this figure indicates the stability boundary of uniform oscillations, yielded by the perturbation analysis (cf. Section 3). Nonlinear effects, i.e. interactions between the modes, lead, however, to the hysteresis phenomena. Therefore the uniform oscillations first appear, when going from the region with a weak feedback, only when the border of the region $\Omega_0$, shown by the dot line in Fig. 16, is crossed.
Stable hexagonal cells are observed in the interval between the curves connecting symbols + and * in Fig. 16. The cells appearing at $\chi = -0.6\pi$ are shown at three subsequent time moments in Fig. 17(a) where the spatial distribution of the oscillation amplitude modulus $\rho$ is displayed in gray scale. We see that in the membranes forming boundaries between the cells the oscillation amplitude is increased, as it can be expected for the shocks. Note also that the structure formed by the spatial distribution of $\rho$ is almost stationary. The linear size of the cells approximately corresponds to the wavelength $\lambda_0$ of the first unstable mode, as yielded by the linear stability analysis (cf. Fig. 3). When the phase shift is increased to $\chi = -0.2\pi$, the cells get larger (Fig. 17(b)).

Inside the narrow region BH in Fig. 16, breathing cellular structures are observed. In such a structure, sizes of the cells change periodically with time, as shown in Fig. 17(c). These structures are stable. Note that the width of this region becomes very small for larger phase shifts.

On the left-hand side of the synchronization window, the hysteresis becomes weaker and approximately at the point $\chi_- = -0.6\pi$ in Fig. 3 the existence boundary of hexagonal cells merges, within the simulation accuracy, with the stability boundary of uniform oscillations. Further to the left from this point, the cells are replaced by stripes (Fig. 17(d)). One of the three modes, needed to produce a cellular structure, begins to dominate and eliminates here the two other modes. This process has analogs in the investigations of stationary Turing structures in reaction–diffusion systems [16].

On the right-hand side of the synchronization window, the cell sizes increase as the point $\chi^* = 0.275\pi$ is approached and become comparable with the total size of the medium. Large phase domains are observed near the stability boundary of uniform oscillations inside the interval $\chi_+ < \chi < \chi^*$ where $\chi_+ = 0.599\pi$. Note that the boundaries separating different regions in the phase diagram of Fig. 16 get closer one to another when the phase shift is increased.
8. Intermittency and turbulent bubbles

Figs. 18(a)–(d) shows examples of intermittent and localized amplitude turbulence in CGLE with global feedback. Each horizontal row in this figure represents a time sequence of frames taken for a different value of the phase shift $\chi$ at a fixed intensity of the feedback. The spatial distribution of the modulus $\rho$ of the local oscillation amplitude is displayed in gray scale, with darker regions corresponding to smaller values of the modulus.
Fig. 18. Intermittent and localized turbulence at $e = 2, \beta = -1.4$ for (a) $\chi = -0.6\pi$ and $\mu = 0.4$, (b) $\chi = -0.4\pi$ and $\mu = 0.25$, (c) $\chi = -0.2\pi$ and $\mu = 0.18$, (d) $\chi = 0$ and $\mu = 0.3$.

Cells in Fig. 18(a) are small and form an almost regular hexagonal array, similar to a steady cellular structure. Some of the cells have now a dark interior, i.e. the modulus of the oscillation amplitude is decreased there. However, the modulus $\rho$ is only reduced but does not vanish. When the spatial phase distribution is plotted for such a pattern, we see that the phase is flipped inside active cells in respect to the rest of the medium. Thin bright membranes, that separate the cells and are formed by shocks, are very rigid. They keep their shape even when a cell becomes active. The entire process can be described as random sporadic activity on a regular array. Individual cells suddenly ‘flip’ and become dark. They remain in this dark state for some time and then return to the normal lightly gray state.

Sometimes the activation destroys a membrane and penetrates into a neighboring cell, infecting it. This process may lead to formation of turbulent aggregates (such an object is present in the third frame of Fig. 18(a)). However, as seen in our simulations, these aggregates later die out and the regime with randomly distributed individual active cells is recovered.
Since we did not run our simulations for very long times, it is impossible to say whether the regime with single active cells is fully persistent or it is eventually replaced by developed defect turbulence when a large turbulent aggregate appears. However, if such a transition occurs, its waiting time should be large.

When the phase shift $\chi$ is increased, cells are getting larger but their arrangement is still fairly regular (Fig. 18(b)). The fraction of active dark cells is now greater and they form clusters or chains. This state is, however, only a transient which is obtained if we start from the initial conditions that represent a hexagonal array. After some time, the pattern loses its regularity and becomes similar to the patterns shown in Fig. 18(c), where a larger value of the phase shift is taken.

In this turbulent state, no regular array of cells is observed. The majority of cells are passive and form a background for the active cells that contain amplitude defects. Such active cells have larger sizes. If two neighboring active cells appear, the membrane between them can sometimes break. Inside an active cell, the modulus distribution is far from uniform and undergoes irregular variation with time.

Passive cells, though irregular, are relatively rigid in the presence of global feedback. Light membranes, forming their boundaries and representing shocks, do not easily break and the seeds of amplitude turbulence do not grow to large sizes. Often, the amplitude defects die out in a seed region and it disappears. New active cells can appear as amplitude defects spontaneously develop inside some of the passive cells. The entire regime can be described as intermittent amplitude turbulence.

At still larger values of the phase shift, not shown in Fig. 16, localized turbulence, which represents individual turbulent islands surrounded by the area with uniform oscillations, is found. An example showing the appearance of such turbulence is given in Fig. 18(d).

Localized turbulence develops through the intermittent amplitude turbulence as the intensity of global feedback is increased. At sufficiently high intensities, lying above the stability boundary of uniform oscillations, passive cells in the background of the intermittent turbulence begin to fade away and then disappear. Active cells, populated by amplitude defects, are, however, more persistent and continue to exist even when they are surrounded by uniform oscillations. As the neighboring passive cells disappear, the shock membrane separating an active cell acquires an approximately circular shape, so that this object looks more like a ‘bubble’. Apparently, this membrane tends to minimize its total length, thus possessing something similar to the elasticity. These bubbles can form clusters, but individual bubbles are also observed.

Fig. 19 displays the spatial distribution of $\cos \phi$ at subsequent time moments within one oscillation cycle when a single turbulent island is present in the medium. To indicate the presence of amplitude defects, we have additionally shown as bright light areas the regions where oscillation amplitude modulus is very low ($\rho < 0.05$). Periodic boundary conditions have been used in this simulation.

Inside a turbulent island, individual cells are seen which merge or split as time goes on. A cell is surrounded by a shock membrane that separates it from the region with uniform oscillations or from other cells in the island. In the center of a cell, the oscillation amplitude sometimes suddenly drops down almost to zero and an approximately concentric wave – an extended amplitude defect – is produced. This wave, which can be described as a string loop, propagates towards the boundary of the cell and dies there out. However, it can also destroy the boundary and penetrate into a neighboring cell.

When the phase variation in an active cell is followed, we see that the phase undergoes rapid variation across the spreading wave, inside the interval characterized by the greatly decreased modulus of the oscillation amplitude. Qualitatively, the pattern looks as if a source emitting ring-shaped phase flips were located in the center of the cell.

These ring-shaped amplitude defects – string loops – show a transverse instability. As they spread out and come closer to the cell’s membrane, some parts of them get very thin. Break-up of such extended defects also often occurs and looks very much similar to breakup of a phase-flip wave shown in Fig. 7. Closer to the BF boundary, where the
Fig. 19. Localized turbulence ($\epsilon = 2, \beta = -1.2, \chi = 0$ and $\mu = 0.33$). Spatial distribution of $\cos \phi$ is shown in gray scale at subsequent time moments within a single oscillation period (from the top left to the bottom right); small bright regions additionally indicate the areas where $\rho < 0.05$.

If a break-up of a string loop has not occurred in a given cell, the cell usually dies out after a few concentric waves have been generated (this is similar to the last stage of destruction of a spiral wave). Apparently, only the broken amplitude defects, which are associated with phase singularities, can produce internal 'pressure' that would allow a cell to counteract the influence of uniform oscillations.

Another representation of localized turbulence is chosen in Fig. 20. As the vertical coordinate, we plot here the inverse $1/\rho$ of the oscillation amplitude modulus (with a cut-off above a certain high level of $1/\rho$). The local gray-scale shading is used to display the local variable $\cos \phi$. The horizontal plane in this figure corresponds to uniform oscillations where $\rho \approx 1.12$. Subsequent time frames within a single oscillation cycle (from the top left to the bottom right) are shown.

The spatial distribution of the oscillation amplitude inside the island is highly irregular. The amplitude reaches very small values in the points that are not fixed and move inside the cells. Oscillations inside a cell are shifted by about half of the period from uniform oscillations in the rest of the medium. However, the phase distribution inside a cell is far from uniform. Great phase variations are seen in the areas where the oscillation amplitude is smaller. Though we have not performed any special quantitative investigation of this complex process, it is, most probably, chaotic.
If no-flux boundary conditions were applied, a turbulent island tended to move into one of the corners of the systems, as in Fig. 20. Once this position was reached, the island could maintain its operation for a long time. Under periodic boundary conditions the lifetime of a turbulent island was significantly shorter.

To investigate the influence of small medium inhomogeneities, special numerical simulations was performed. A local inhomogeneity was introduced by assuming that inside a small region in the center of the medium the oscillation frequency $\omega$ in CGLE was slightly increased by amount $\delta \omega$ in comparison to the rest of the medium. Periodic boundary conditions were used in these simulations.

We have found that even such small medium imperfections, which were not noticeable on the background of uniform oscillations, could serve as nucleation centers and pin the turbulent islands. Fig. 21 shows formation of a small active bubble on the background of a cellular structure. Turbulent islands could also be pinned on the background of uniform oscillations.

Fig. 22 shows time evolution of the synchronization parameter $R$ when we started from the turbulent state without global feedback and then abruptly switched it on in absence (thick line) and in the presence (thin line) of a small inhomogeneity. We see that, although in the homogeneous medium such strong global feedback soon leads to the
Fig. 21. Development of an active pinned bubble on the background of the cellular structure ($\varepsilon = 2.0, \beta = -1.4, \chi = -0.2\pi, \mu = 0.32$); the modulus $\rho$ of the local oscillation amplitude is plotted. Inside the defect of size $5 \times 5$, located in the center of the medium, the local oscillation frequency is increased by $\delta \omega = 0.35$.

Fig. 22. Turbulence induced by a small defect of size $2 \times 2$, inside which the local oscillation frequency is increased by $\delta \omega = 0.05 (\varepsilon = 2, \beta = -1.2, \chi = 0)$.

emergence of synchronous oscillations, even a weak inhomogeneity is sufficient to break the synchronization down and produce localized turbulence in the medium.

9. Discussion

Global feedback essentially influences properties of turbulence in the two-dimensional complex Ginzburg–Landau equation and leads to appearance of new kinds of patterns. Inside a synchronization window, strong feedbacks
suppress turbulence and establish uniform oscillations. The action of global feedback depends on its intensity and the phase shift, controlled by the delay time.

By adjusting the delay, the synchronization window can always be reached at sufficiently high feedback intensities. Synchronization develops according to three different scenarios, i.e. via formation of hexagonal cellular structures or stripes, via localized turbulence on the background of uniform oscillations, and via formation of large phase domains. Strong global feedback can destroy individual spiral waves. Two kinds of elementary structures play a significant role in these processes, i.e. shocks and phase flips.

Our theoretical investigations have been carried out for a concrete model of CGLE with global feedback. However, this model is generic since it describes the typical behavior of a reaction–diffusion system near a supercritical Hopf bifurcation. We expect that it would also qualitatively describe the behavior found farther away from the bifurcation point where oscillations are not nearly harmonical and their amplitudes are not small.

Acknowledgements

We acknowledge the financial support of the Volkswagen–Stiftung, and are grateful to G. Dewel and R. Imbihl for valuable discussion.

References